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MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Twenty-second Summer Meeting, Madison, Wis., September 4–7, 1939.
Twenty-fourth Annual Meeting, Columbus, Ohio, December 26–30, 1939.

The following is a list of the Sections of the Association, with dates of those Section meetings which have been scheduled for 1939 and reported to the Secretary.

Allegheny Mountain, May 13.
Illinois, Galesburg, May 12–13.
Indiana, Muncie, April 28–29.
Iowa, Ames, April 21–22.
Kansas, Topeka, April 1.
Kentucky.
Louisiana-Mississippi, Baton Rouge, La., March 3–4.
Maryland-District of Columbia-Virginia, May 13.
Michigan, Ann Arbor, March 18.
Minnesota.
Missouri, Springfield, April 28.
Nebraska, Lincoln, May 5.
Ohio, Columbus, April 8.
Oklahoma, Tulsa, February 10.
Philadelphia, Bethlehem, Pa., December 2.
Rocky Mountain, Laramie, Wyo., April 28–29.
Southeastern, Charleston, S.C., March 24–25.
Southern California, Whittier, March 4.
Southwestern, Alpine, Texas, May 2–3.
Texas, Abilene, March 31–April 1.
Wisconsin, Milwaukee, May 6.

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SOME TOPICS IN MATHEMATICAL STATISTICS

J. F. KENNEY, Northwestern University

1. Fundamental notions and definitions. First courses in statistics deal largely with descriptive characterizations of quantitative data. When N observations on one or more variables are available, such characterizations consist in computing the numerical values of certain functions, such as the mean, variance, and correlation coefficient, denoted respectively by $\bar{x}$, s^2, and r, and defined as follows:

$$N\bar{x} = \sum x_i, \qquad Ns_x^2 = \sum (x_i - \bar{x})^2, \qquad Nrs_xs_y = \sum x_iy_i - N\bar{x}\bar{y}.$$

(The sign $\sum$ here and hereafter indicates in every case summation over values of the index from 1 to N.)

Both for practical and for interesting theoretical purposes it is important to construct a theory dealing with abstract notions. This theory, in the field of statistics, is commonly called mathematical statistics. It is an idealization of observed distributions and their summarizing functions, comparable to the idealization of the outlines of material objects into the figures of geometry.

The main object of the present paper is to give, in brief form, a nearly self-contained exposition of certain developments in this field that are not so well known as they should be to many teachers of college courses in statistics.

A fundamental notion in mathematical statistics is the concept of distribution function. It relates to a *population* or universe of discourse. A continuous variable x is said to have the distribution function $f(x)$, which we take to be single-valued and non-negative, if the frequency of occurrence of x in the range $a<x<b$ is measured by

$$\int_a^b f(x)dx. \tag{1}$$

A constant factor in the distribution function may be determined so that

$$\int_{-\infty}^{\infty} f(x)dx = 1, \tag{2}$$

and under this condition the integral (1) denotes the probability that x lies in the interval (a, b). If the actual occurrence of the variable is limited to a finite range, $f(x)$ is defined to be identically zero outside that range.

Corresponding definitions will now be given for a distribution function of two variables. The continuous variables (x, y) have the joint distribution function $f(x, y)$ if the double integral of $f(x, y)$ over a region of the (x, y)-plane measures the frequency of occurrence of pairs of values (x, y) in that region. It will be understood that $f(x, y)$ is single-valued and non-negative. If values of (x, y) are restricted to a finite region we define $f(x, y)$ to be identically zero outside that region. In the extended region of definition, if

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)dydx = 1 \tag{3}$$

then

$$\int_{a}^{b}\int_{c}^{d} f(x, y)dydx \tag{4}$$

represents the probability that x lies between a and b at the same time that y lies between c and d.

In the case of joint distribution functions we distinguish between two cases: (a) when the variables are independent in the probability sense, and (b) when they are correlated. The variables x and y are independent in the probability sense if

$$f(x, y) \equiv g(x)h(y)$$

where $g(x)$ and $h(y)$, called marginal distributions, are defined as follows: $g(x)=\int f(x, y)dy$ and $h(y)=\int f(x, y)dx$, the integrals being taken over all admissible values of the variables. When x and y are correlated, $f(x, y)$ cannot be expressed as the product of the functions describing their distributions separately.

2. Expected values. Let the variable x be subject to the distribution function $f(x)$ and let $\phi(x)$ be an arbitrary function of x. Then the expected value of $\phi(x)$, denoted by application of the operator E, is defined by

$$E[\phi(x)] = \int_{-\infty}^{\infty} \phi(x)f(x)dx$$

provided this integral exists. In particular, if $\phi(x)=x^k$, $(k=1, 2, \cdots)$, we have

$$E(x^k) = \int_{-\infty}^{\infty} x^k f(x)dx. \tag{5}$$

For $k=1$, (5) defines the mean of the x's in the universe represented by $f(x)$. This will be denoted by $\tilde{x}$ in contradistinction to the mean, $\bar{x}$, of a sample. Therefore we may write

$$E(x) = \tilde{x}. \tag{6}$$

If $\phi(x)=(x-\tilde{x})^2$, we have the variance of x

$$\begin{aligned} \sigma_x^2 &= E(x-\tilde{x})^2 \\ &= E(x^2) - \tilde{x}^2. \end{aligned} \tag{7}$$

The (positive) square root of σ_x^2 is called the standard deviation or standard error of the distribution of x. Analogous definitions hold, of course, for y.

If the variables x and y are simultaneously distributed in accord with the function $f(x, y)$, then

$$E(xy) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf(x, y)dydx.$$

The correlation coefficient, ρ, between x and y in the bivariate universe represented by $f(x, y)$ is defined by

$$\rho\sigma_x\sigma_y = E(xy) - \tilde{x}\tilde{y}. \tag{8}$$

The quantities $\tilde{x}$, σ, ρ, and so on, relating to a universe, are called parameters.

The following propositions may easily be established, and so they are stated without proof.

I. *The expected value of the product of a variable and a constant is equal to the product of the constant and the expected value of the variable. That is,*

$$E(cx) = cE(x).$$

II. *The expected value of deviations of a variable from its expected value is zero. That is,*

$$E(x - \tilde{x}) = 0.$$

III. *The expected value of the sum of two or more variables is the sum of their expected values. In symbols,*

$$E(x + y + z) = E(x) + E(y) + E(z).$$

IV. *If x and y are mutually independent in the probability sense, then the expected value of their product is equal to the product of their expected values. That is,*

$$E(xy) = E(x)E(y).$$

V. *The expected value of the product of deviations of two mutually independent variables from their expected values is zero. That is,*

$$E[(x - \tilde{x})(y - \tilde{y})] = 0.$$

VI. *The expected value of the product of deviations of two correlated variables from their expected values is given by*

$$E[(x - \tilde{x})(y - \tilde{y})] = \rho\sigma_x\sigma_y.$$

3. Standard error of a linear function of variables. Let w be a linear function defined by

$$w = c_1x_1 + c_2x_2 + \cdots + c_Nx_N, \tag{9}$$

where each variable x_k, $(k = 1$ to $N)$, is arbitrarily distributed and where the c's are arbitrary constants. Let σ_k represent the standard error of x_k in the universe to which it belongs, σ_w the standard error of the distribution of w, and ρ_{ij} the correlation coefficient (if any correlation exists) between x_i and x_j. We seek the standard error of w in terms of σ_k and ρ_{ij}, $(i = 1$ to $N, j = 1$ to $N)$.

Case I. We will suppose first that the variables in the several universes are correlated, that is, that ρ_{ij} is different from zero for every combination of i and j. From (9) and Proposition III we have

$$E(w) = c_1E(x_1) + c_2E(x_2) + \cdots + c_NE(x_N), \tag{10}$$

that is,

$$\tilde{w} = c_1\tilde{x}_1 + c_2\tilde{x}_2 + \cdots + c_N\tilde{x}_N. \tag{11}$$

Then

$$E(w - \tilde{w})^2 = \sum c_i^2 E(x_i - \tilde{x}_i)^2 + \sum_{i \neq j} c_ic_jE(x_i - \tilde{x}_i)(x_j - \tilde{x}_j),$$

which by definition (7) and Proposition VI becomes

$$\sigma_w^2 = \sum c_i^2\sigma_i^2 + \sum_{i \neq j} c_ic_j\rho_{ij}\sigma_i\sigma_j. \tag{12}$$

If $c_1 = 1$, $c_2 = \pm 1$, and $N = 2$, we have as a special case

$$\sigma_w^2 = \sigma_1^2 \pm 2\rho_{12}\sigma_1\sigma_2 + \sigma_2^2. \tag{13}$$

Case II. Suppose the x's in (9) are mutually independent in the statistical sense so that $\rho_{ij} = 0$. Then (12) becomes

$$\sigma_w^2 = c_1^2\sigma_1^2 + c_2^2\sigma_2^2 + \cdots + c_N^2\sigma_N^2. \tag{14}$$

4. Theorems. Relations (10)–(14) enable us to prove some interesting and useful theorems about the distribution of means of samples from an arbitrary universe. Let $(x_1, x_2, \cdots, x_N)$ be a set of N independent variables each subject to the same distribution function g, so that their joint distribution function is

$$f(x_1, x_2, \cdots, x_N) \equiv g(x_1)g(x_2) \cdots g(x_N).$$

Then $(x_1, x_2, \cdots, x_N)$ is called a *sample* of N from a universe with distribution function $g(x)$.

THEOREM I. *If samples of size N be drawn from an arbitrary universe and if $\bar{x}$ be the mean of a sample, then the mean of all possible such means equals the mean of the universe. That is,* $E(\bar{x}) = \tilde{x}$.

Proof. In (9), let $c_1 = c_2 = \cdots = c_N = 1/N$ and let $x_1, x_2, \cdots, x_N$, constitute a sample from a universe with mean x and variance σ_x^2. Then $w = \bar{x}$. As a consequence of the definition of sample, $E(x_i) = \tilde{x}$ for each value of i from 1 to N. Therefore (10) becomes, $E(\bar{x}) = \tilde{x}$.

THEOREM II. *The variance of the distribution of means of samples of size N from an arbitrary universe equals the variance of the universe divided by N. In symbols,*

$$\sigma_{\bar{x}}^2 = \sigma_x^2/N. \tag{15}$$

Proof. As in the proof of Theorem I let $w = \bar{x}$. Then (14) becomes

$$\sigma_{\bar{x}}^2 = (\sigma_1^2 + \sigma_2^2 + \cdots + \sigma_N^2)/N^2. \tag{16}$$

Since the x's constitute a sample, $\sigma_i^2 = \sigma_x^2$ for each value of i from 1 to N. So (16) reduces to (15).

In the next three theorems it will be understood that x and y are correlated variables which are simultaneously distributed in accord with $f(x, y)$ in which the parameters are $\tilde{x}$, $\tilde{y}$, σ_x, σ_y, and ρ.

THEOREM III. *Let (x_1, y_1), (x_2, y_2), $\cdots$, (x_N, y_N) be a sample of N pairs drawn independently from $f(x, y)$ and let $(\bar{x}, \bar{y})$ be the mean of a sample. The correlation coefficient, R, between the means of all possible such samples equals ρ.*

Proof. By definition

$$R = \frac{E(\bar{x}\bar{y}) - \tilde{x}\tilde{y}}{\sigma_{\bar{x}}\sigma_{\bar{y}}}\cdot \tag{17}$$

And

$$\begin{aligned} E(\bar{x}\bar{y}) &= \frac{1}{N^2} E[(x_1 + x_2 + \cdots + x_N)(y_1 + y_2 + \cdots + y_N)] \\ &= \frac{1}{N^2} E(S), \end{aligned}$$

where

$$\begin{aligned} S = {} & x_1y_1 + x_1y_2 + \cdots + x_1y_N \\ & + x_2y_1 + x_2y_2 + \cdots + x_2y_N \\ & + \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ & + x_Ny_1 + x_Ny_2 + \cdots + x_Ny_N. \end{aligned}$$

We will separate S into two parts, conveniently called u and v, where

$$u = x_1y_1 + x_2y_2 + \cdots + x_Ny_N,$$

and

$$v = \text{sum of } (N^2 - N) \text{ terms of the form } x_iy_j, \qquad i \neq j.$$

Then

$$E(u) = E[\textstyle\sum (x_iy_i)] = \sum [E(x_iy_i)] = NE(xy).$$

In v, x_i must be uncorrelated with y_j since $i \neq j$. Therefore

$$E(x_iy_j) = E(x_i)E(y_j) = \tilde{x}\tilde{y},$$

and

$$E(v) = (N^2 - N)\tilde{x}\tilde{y}.$$

So we have

$$E(S) = NE(xy) + (N^2 - N)\widetilde{x}\widetilde{y},$$

and therefore

$$E(\bar{x}\bar{y}) = \frac{1}{N}\left[E(xy) + (N-1)\widetilde{x}\widetilde{y}\right]. \tag{18}$$

Making use of Theorem II and (18) the right member of (17) reduces to the definition of ρ.

THEOREM IV. *Let $\bar{x}$ be the mean of a sample of N from $g(x)$ and let $\bar{y}$ be the mean of a sample of N from $h(y)$ where $g(x)$ and $h(y)$ are the marginal distributions of the universe $f(x, y)$ of correlated variables. Let $w = \bar{x} - \bar{y}$. The variance of the sampling distribution of w is*

$$\sigma_w^2 = \frac{1}{N}\left\{\sigma_x^2 - 2\rho\sigma_x\sigma_y + \sigma_y^2\right\}. \tag{19}$$

The proof follows from (13) and Theorem III.

THEOREM V. *Let $\bar{x}$ and $\bar{y}$ be the means, s_x and s_y the standard deviations, and r the correlation coefficient of a sample of N correlated items. Suppose N is so large that s^2 is a good estimate of σ^2 and r of ρ, so that we may write*

$$\sigma_{\bar{x}}^2 = s_x^2/N, \qquad \sigma_{\bar{y}}^2 = s_y^2/N, \qquad \rho = r.$$

The variance of the sampling distribution of w, referred to in Theorem IV, may be computed from the sample by the formula

$$\sigma_w^2 = \frac{1}{N^2}\left[\sum (x_i - y_i)^2 - \frac{(\sum x_i - \sum y_i)^2}{N}\right]. \tag{20}$$

The proof follows from (19).

5. Expected value of sample variances. The definition of the variance, s^2, of a sample may be written

$$s^2 = (x_1^2 + x_2^2 + \cdots + x_N^2)/N - \bar{x}^2.$$

Then the expected value of s^2 from repeated samples is

$$E(s^2) = E\left[\frac{1}{N}(x_1^2 + x_2^2 + \cdots + x_N^2)\right] - E(\bar{x}^2).$$

Since the x's constitute a sample, we may write

$$E(x_1^2 + x_2^2 + \cdots + x_N^2) = NE(x^2),$$

and from (18), replacing y by x there, we have

$$E(\bar{x}^2) = \frac{1}{N}\left[E(x^2) + (N-1)\widetilde{x}^2\right].$$

Therefore

$$E(s^2) = \frac{1}{N}\,[NE(x^2)] - \frac{1}{N}\,[E(x^2) + (N-1)\tilde{x}^2]$$

$$= \frac{N-1}{N}\,[E(x^2) - \tilde{x}^2],$$

and using (7) we obtain

$$E(s^2) = \frac{N-1}{N}\,\sigma^2. \tag{21}$$

Thus the mean of the sampling distribution of s^2 from an arbitrary universe is given by (21).

As Jackson [1] observes, it is to be anticipated that the expected value of s^2 is less than σ^2. The variance σ^2 refers to deviations from $\tilde{x}$, whereas any s^2 refers to deviations from an $\bar{x}$. For any sample, then, we may regard $\tilde{x}$ as an arbitrary origin. Since, in the case of any sample, the sum of the squares of deviations from its mean, $\bar{x}$, is less than the sum of the squares of the deviations of the same variates from an arbitrary point $\tilde{x}$ (unless the sample is one whose mean falls at $\tilde{x}$), it is to be expected that the mean of all the values of s^2 will be less than σ^2. Relation (21) measures the extent of this inequality.

6. Unbiased estimates of population parameters. A distribution function is not only a function of the variable involved, but it is also a function of the parameters, or hypothetical quantities, which are introduced to specify the universe sampled. For example, in the normal distribution function

$$f(x) = \frac{1}{(2\pi)^{1/2}\sigma}\, e^{-(x-\tilde{x})^2/2\sigma^2}$$

the population mean $\tilde{x}$ and the variance σ^2 are the parameters.

An appropriate function of the variates given by a sample for estimating a parameter is called a *statistic*. If the expected value of a statistic equals the corresponding parameter, then the statistic is called an *unbiased estimate* of the parameter.

It is clear from Theorem I that $\bar{x}$ is an unbiased estimate of $\tilde{x}$. Let $\dot{\sigma}^2$ be an unbiased estimate. If this estimate* is based on a single sample, then from (21) we have

$$\dot{\sigma}^2 = \frac{N}{N-1}\, s^2 = \frac{\sum (x_i - \bar{x})^2}{N-1}. \tag{22}$$

If $n = N-1$ it is obvious that

$$s^2 = \frac{n}{n+1}\,\dot{\sigma}^2. \tag{22a}$$

---

* R. A. Fisher and some other writers use the symbol s^2 to denote this unbiased estimate of σ^2.

It is conventional [**2, 3**] to take

$$\dot{\sigma} = \left\{\frac{N}{N-1}\right\}^{1/2} s \tag{23}$$

as an estimate of σ. If N is large the difference between unity and the coefficient of s in (23) is negligible in numerical problems. With N large it would not be invalid, to any appreciable extent, to use s as an estimate of σ.

If two independent samples are available from the same universe, an unbiased estimate based on the two samples is given by

$$\dot{\sigma}^2 = q/(N-2), \tag{24}$$

where

$$q = N_1 s_1^2 + N_2 s_2^2, \qquad N = N_1 + N_2,$$

s_1^2 and s_2^2 being the variances of samples consisting of N_1 and N_2 variates respectively.

In case k independent samples are available from the same universe, we may generalize (24) and write

$$\dot{\sigma}^2 = Q/(U-k), \tag{25}$$

where

$$Q = N_1 s_1^2 + N_2 s_2^2 + \cdots + N_k s_k^2,$$
$$U = N_1 + N_2 + \cdots + N_k,$$

and s_i^2 is the variance of the ith sample consisting of N_i variates. In subsequent references to $\dot{\sigma}$ it will be clear from the context, if it is not stated explicitly, whether this estimate is based on 1, 2, or k samples.

If $N_i = N$ is the same for every sample, (25) reduces to

$$\dot{\sigma}^2 = \frac{N(s_1^2 + s_2^2 + \cdots + s_k^2)}{U-k}, \tag{26}$$

where $U = Nk$. Clearly, (26) may be written in the form

$$\frac{N-1}{N}\dot{\sigma}^2 = \frac{1}{k}(s_1^2 + s_2^2 + \cdots + s_k^2). \tag{26a}$$

When k is taken infinitely large so that U becomes the universe, the right member of (26a) then refers to the expected value of s^2 and $\dot{\sigma}^2$ becomes σ^2 itself. Hence as $k \to \infty$ the limiting value of (26a) becomes

$$\frac{N-1}{N}\sigma^2 = E(s^2),$$

as given in (21).

As an alternate to (26), in the case where all samples contain the same number of variates, we may take

$$\dot{\sigma} = \frac{1}{b(N)} \times \frac{1}{k} (s_1 + s_2 + \cdots + s_k) \tag{27}$$

$$= \frac{1}{b(N)} \times \text{mean value of standard deviations,}$$

where $b(N)$ is a function of N which approaches unity as N increases. The exact expression for $b(N)$ will be derived in §9. We have as an approximate value $b(N) = 1 - 3/4N$. The limiting value of (27) as $k \to \infty$ is

$$\sigma = E(s)/b(N). \tag{28}$$

In §9 we shall show that $b(N)\sigma$ is the expected value of the sampling distribution of s from a normal universe whose standard deviation is σ. Values of $b(N)$ have been tabulated by E. S. Pearson [**4**] and others [**5**].

As an alternate to (23) we have from (27) when $k = 1$,

$$\dot{\sigma} = s/b(N). \tag{29}$$

7. Degrees of freedom. In §6 we have proved, essentially, that the expected value of $\Sigma(x_i - \bar{x})^2$ is $(N-1)\sigma^2$, where the N values of x in the sample are subject to the linear restriction $\Sigma x_i = N\bar{x}$. This is equivalent to proving that the expected value of Σx_i^2 is $(N-1)\sigma^2$ when the x's are subject to the linear restriction $\Sigma x_i = 0$. Suppose, however, that there are $k < N$ linear restrictions on the x's. What then is the expected value of Σx_i^2? A. T. Craig [**6**] has proved analytically that if $x_1, x_2, \cdots, x_N$ are N independent values of a variable which is normally distributed about zero with variance σ^2 and if the N values of x are subject to $k < N$ homogeneous linear restrictions, then the expected value of Σx_i^2 is $(N-k)\sigma^2$. The number $n = N - k$ is frequently called the number of "degrees of freedom."

8. "Student's" distribution. We have shown (Theorems I and II) that the means of samples of N from an arbitrary universe with mean $\tilde{x}$ and variance σ^2 are distributed about $\tilde{x}$ with variance σ^2/N. It is well known and can readily be proved (see below) that if the universe is normal then the means are themselves normally distributed. In fact, if the x's in (9) are independent and normally distributed, then w is normally distributed. This property of a linear function of normally distributed variables being normally distributed is sometimes called the *reproductive property* of the normal law.

On the side of applications, σ^2 is seldom available and must be estimated from the available data. If we use the estimate defined in (22) we may replace σ^2/N by $s^2/(N-1)$. In testing a null [**7**] hypothesis that a sample of N comes from a proposed normal universe, using the difference between the observed and hypothetical means as a criterion of judgment, it had long been the custom to refer the calculated value of

$$t = \frac{\bar{x} - \tilde{x}}{s/\sqrt{N-1}} \tag{30}$$

to a normal probability scale. While Helmert obtained the distribution of s^2 as early as 1876, it seems that "Student" [**8**] was the first (1908) to recognize the importance, when N is not large, of taking account of the variability of s in (30). By means of a remarkable intuition he obtained, somewhat empirically, the simultaneous distribution function for $\bar{x}$ and s from a normal universe. Later writers, notably Fisher [**9**], established his results rigorously and proved that

$$f(\bar{x}, s) = g(\bar{x})h(s), \tag{31}$$

where

$$g(\bar{x})d\bar{x} = k_1 e^{-N(\bar{x}-\tilde{x})^2/2\sigma^2} d\bar{x}, \qquad -\infty \leqq \bar{x} \leqq \infty,$$

$$h(s)ds = k_2 e^{-Ns^2/2\sigma^2} s^{N-2} ds, \qquad 0 \leqq s \leqq \infty,$$

$$k_1 = (2\pi\sigma^2/N)^{-1/2}, \qquad k_2 = \frac{2\left(\dfrac{N}{2\sigma^2}\right)^{(N-1)/2}}{\Gamma\left(\dfrac{N-1}{2}\right)}.$$

From these results it is easy to show that (30) is distributed in accordance with what has come to be called the "Student" curve

$$F_n(t) = K_n(1 + t^2/n)^{-(n+1)/2}, \tag{32}$$

where $1/K_n = n^{1/2}B(n/2, 1/2)$, B being the Beta function and n the number of degrees of freedom in the estimate of σ^2. It is important to observe that the criterion (30) and its probability function (32) are completely expressible in terms of the observations. Such criteria have been called by Wilks "Studentized" functions. Tables [**10**] of the probability $P = 1 - P_n(t)$, where

$$P_n(t) = 2\int_0^t F_n(t)dt, \tag{33}$$

are available for assigned values of n and t.

Fisher showed (*loc. cit.*) that the "Student" curve has a much wider range of applications than the problem for which it was designed. He demonstrated that (32) can be used in testing the significance of the difference between two sample means and in testing the significance of regression coefficients and of certain curvilinear regressions. The scheme by which the "Student" curve is made available to these other problems consists in building a variable in the nature of a fraction

$$t = \frac{v}{\dot{\sigma}_v} \tag{34}$$

whose numerator is a statistic normally distributed about zero and whose denominator is an independently distributed and unbiased estimate of the standard deviation of the statistic in the numerator. It is imperative that the numerator and denominator be independent.

As an example of (34), let $v=\bar{x}_1-\bar{x}_2$ be the difference between the means of two independent samples consisting of N_1 and N_2 items, respectively, from a normal universe with variance σ^2. Then we have

$$\dot{\sigma}_v = \dot{\sigma}\left[\frac{N_1 + N_2}{N_1N_2}\right]^{1/2},$$

where $\dot{\sigma}$ is defined in (24), and (34) is distributed in accord with (32) for $n=N_1+N_2-2$. In this connection, the writer [11] has called attention to a more appropriate formula than the one frequently given in textbooks for $\dot{\sigma}_v$ when the samples are large.

Recently, Rider [12] and Rietz [13] have given excellent surveys of the contributions of "Student," Fisher, and others to the theory of sampling. Jackson (*loc. cit.*) has explained, at the level of "ordinary courses and textbooks in mathematics," how the establishment of "Student's" results by Fisher, which involves N-dimensional geometry, can be put in completely analytical form.

9. The distribution of ***s*****.** The distribution of the standard deviations of samples of N from a normal universe is given by $h(s)$ in (31). So its mean value is

$$E(s) = \int_0^\infty h(s)sds$$

which upon integration yields

$$E(s) = \frac{(2/N)^{1/2}\Gamma(N/2)\sigma}{\Gamma(\overline{N-1}/2)}. \tag{35}$$

The coefficient of σ in (35) is the function we denoted by $b(N)$ in (28). Romanovsky [14] showed that

$$b(N) \to \left(1 - \frac{3}{4N} - \frac{7}{32N^2} - \cdots\right).$$

The modal value of s, found by making $h(s)$ a maximum, is

$$s = \sigma(\overline{N-2}/N)^{1/2}.$$

Romanovsky deduced the standard deviation of s to be

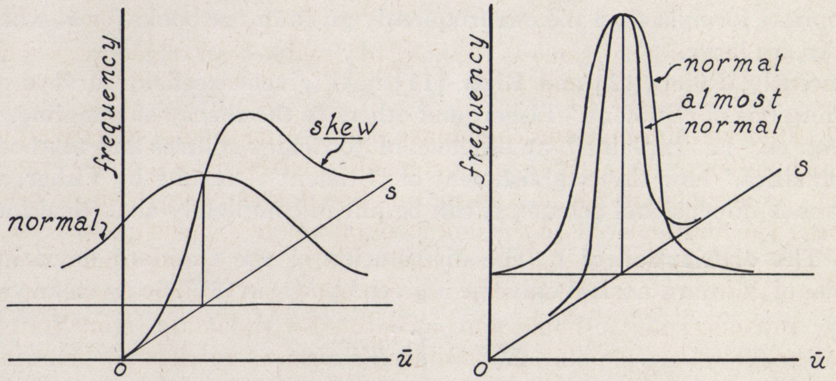
$$\sigma_s = \sigma\left(\frac{1}{2N} - \frac{2}{8N^2} - \frac{3}{16N^3} - \cdots\right)^{1/2}.$$

The approximate value $\sigma_s=\sigma(2N)^{-1/2}$ is often used in practice, and this is the basis for the common statement that the standard error of a standard deviation is $1/\sqrt{2}$ times that of a mean.

10. The $(\bar{x}, s)$-frequency surface. Since

$$\int_0^\infty \int_{-\infty}^\infty f(\bar{x}, s)d\bar{x}ds = 1,$$

the volume under the surface over a closed contour in the $\bar{x}s$-plane represents the proportion or percentage of samples whose means and standard deviations fall simultaneously within the ranges defined by the boundary of the given contour. In depicting this surface it is convenient to let $\bar{u}=\bar{x}-\tilde{x}$, so that the origin of $\bar{u}$ is at $\bar{x}=\tilde{x}$. In a comprehensive paper by Deming and Birge [**15**] two such frequency surfaces are represented. These are reproduced in Figure 1, one for a small value of N and the other for a comparatively large value of N.



N small, about 10 N larger, about 30

FIG. 1. The frequency surface $f(\bar{u}, s)=Ke^{-N(\bar{u}^2+s)/2\sigma^2}$

As the authors point out, the highest point of the surface has the coördinates $\bar{u}=0$, $s=\sigma(\overline{N-2}/N)^{1/2}$. Because of the independence of $\bar{x}$ and s, all plane sections $s=\text{constant}$ will be normal curves with standard deviation equal to $\sigma/\sqrt{N}$. The $\bar{u}=\text{constant}$ sections will be skew curves whose equations are given by $h(s)$. They will all have the same mean and mode. As N increases, their mean and mode approach coincidence with the value σ while the curves lose their skewness and become normal with center at $s=\sigma$ and standard deviation equal to $\sigma/\sqrt{2N}$. As N increases, the surface becomes more and more concentrated about the point $\bar{u}=0$, $s=\sigma$.

11. The s^2 and χ^2 distributions. Since $ds^2=2sds$ the distribution of s^2, which we will denote by $H(s^2)$, can be found at once from $h(s)$. Thus we obtain

$$H(s^2)ds^2 = \frac{e^{-Ns^2/2\sigma^2}(Ns^2/2\sigma^2)^{(N-3)/2}ds^2}{(2\sigma^2/N)\Gamma(\overline{N-1}/2)}. \tag{36}$$

In §5, the mean of the sampling distribution of s^2 from an arbitrary population was obtained. It is interesting to verify that result in the present case where the distribution function is known. The mean of the distribution of variances of samples of N from a normal universe is given by

$$E(s^2) = \int_0^\infty H(s^2)s^2ds^2 = \sigma^2(N-1)/N.$$

The standard deviation of the $H(s^2)$ distribution is, approximately,

$$\sigma_{s^2} = \sigma^2\sqrt{2/N}.$$

If, in (36), we let $\chi^2 = Ns^2/\sigma^2$ we get the χ^2 distribution

$$T_n(\chi^2)d\chi^2 = \frac{e^{-\chi^2/2}(\chi^2/2)^{(N-3)/2}d\chi^2}{2\Gamma(\overline{N-1}/2)}\cdot \tag{37}$$

This χ^2, although not the same as the one [**16**] used in tests of goodness of fit, has essentially the same distribution.

12. **Fiducial inference and confidence limits.** A method of inverse argument by which values of population parameters are inferred from samples "randomly drawn" from populations of known functional form has recently been developed. The limits placed on the unknown parameters in such an inverse argument have been called *fiducial* or *confidence* limits. In arguing from a sample to the population, any inference must be attended with some degree of uncertainty. But uncertainty should not be confused with lack of rigor. Statements can be made about population parameters, subject to risks of being wrong, where the error is precisely expressed in terms of probability theory. In other words, the nature and degree of the uncertainty can be rigorously expressed. The modern method of expressing the reliability of a statistical estimate of a parameter in terms of fiducial limits seems likely to replace the traditional but often misleading method of expression involving "probable error."

(a) *For the mean.* Let $\bar{x}$ and s be the mean and standard deviation of a sample of $N = n+1$ items from a normal universe with unknown mean $\tilde{x}$. The problem is to determine an interval surrounding $\bar{x}$ in which we may assume, with a certain degree of confidence, that $\tilde{x}$ is contained. Suppose we make the claim

$$\bar{x} - t_\alpha s/\sqrt{n} < \tilde{x} < \bar{x} + t_\alpha s/\sqrt{n}.$$

Let the probability of an error in a statement of this sort be equal to or less than α, $0 < \alpha < 1$, α being chosen in advance. The quantities $\bar{x} \pm t_\alpha s/\sqrt{n}$ are the fiducial limits for $\tilde{x}$ corresponding to a given value of α. Our measure of confidence in such a claim is called *fiducial probability.* We can obtain from tables [**10**] the value of t, t_α, corresponding to assigned values of n and α, by assigning to $P = 1 - P_n(t)$ a value of α appropriate to the level of confidence chosen, solving for t, and then determining the limits for $\tilde{x}$ from the relation

$$\pm\, t = (\bar{x} - \tilde{x})\sqrt{n}/s.$$

It is conventional among certain workers to take $\alpha = .01$ since they wish to determine values of $\tilde{x}$ in an interval dividing hypotheses that will be rejected from those acceptable under a null hypothesis at the 1% level of significance.

To illustrate, for $n = 15$, we find from the tables that $t = \pm 2.947$ when $P = .01$. Then we have

$$(\bar{x} - \tilde{x}) = \pm\ 2.947s/\sqrt{15} = \pm\ .76s,$$

and the claim

$$\bar{x} - .76s < \tilde{x} < \bar{x} + .76s$$

will be correct 99% of the time.

It is clear from the above procedure that our confidence in the fiducial limits $\bar{x} \pm t_\alpha s/\sqrt{n}$ is measured by the area under the $F_n(t)$ curve inside $t = \pm t_\alpha$, that is, by $P_n(t_\alpha)$ (Figure 2). This means that if we could observe all possible samples, the proportion represented by $P_n(t_\alpha)$ would yield values of $\bar{x}$ and s for which the claim is true, while the remaining proportion, $P = 1 - P_n(t_\alpha)$ would yield values of $\bar{x}$ and s for which the claim is false.

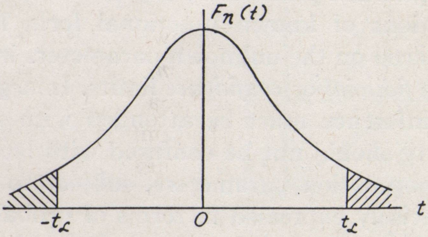


FIG. 2.

If we were testing a hypothetical value of $\bar{x}$ we would say that $\bar{x}$ is *not significant* at the 1% level of significance if $\tilde{x}$ has any value in the $\bar{x} \pm t_\alpha s/\sqrt{n}$ interval, $\alpha = .01$. If $\tilde{x}$ does not lie in this interval we say that $\bar{x}$ is *significant* at this level.

Obviously, values of t satisfying the equation $P = .01$, that is, $P_n(t) = .99$, vary with n. When the sample is large an alternate method may be used which avoids the trouble of entering a table. The variable

$$t = (\bar{x} - \tilde{x})\sqrt{N}/s$$

is approximately normally distributed for $N > 30$. The area under the normal curve outside $t = \pm 2.576$ is .01. Therefore, the 99% fiducial range of $\tilde{x}$ is

$$\bar{x} \pm\ 2.576s/\sqrt{N}$$

and the range gets smaller as N increases.

(b) *For the difference between two means*. Let $\bar{x}_1$ and s_1^2 be the mean and variance of a sample of N_1 from a normal universe with unknown mean $\tilde{x}_1$ and let

$\bar{x}_2$ and s_2^2 be the mean and variance of a sample of N_2 from a normal universe with unknown mean $\bar{x}_2$. It is assumed that the two universes have a common variance σ^2. For brevity, let

$$\overline{w} = \bar{x}_1 - \bar{x}_2, \qquad \tilde{w} = \tilde{x}_1 - \tilde{x}_2, \qquad N = N_1 + N_2,$$

$$\sigma_{\overline{w}} = \left[\frac{N_1 s_1^2 + N_2 s_2^2}{N - 2} \times \frac{N_1 + N_2}{N_1 N_2}\right]^{1/2}.$$

Then

$$t = \frac{\bar{w} - \tilde{w}}{\sigma_{\overline{w}}} \tag{38}$$

is distributed in accord with $F_n(t)$ for $n = N-2$. From (38), upper and lower fiducial values of $\tilde{w}$ can be found by assigning to t the solutions of $P_n(t) = .99$, that is, of $P = .01$. If the value $\tilde{w} = 0$ falls outside the fiducial interval thus established, the conclusion is that the difference between the means is significant at the 1% level. That is, $\tilde{w} \neq 0$ and hence $\tilde{x}_1 \neq \tilde{x}_2$.

If the two samples are equal in number so that the variates can be paired in some way we may compute (38) by a different method. Let $N = N_1 = N_2$, $w = x_1 - x_2$, and compute $\overline{w}$ and $\Sigma(w_i - \overline{w})^2$. Then

$$t = \frac{\bar{w} - \tilde{w}}{\sigma_w} = \frac{\bar{w} - \tilde{w}}{s_w/\sqrt{N - 1}}$$

$$= \frac{\bar{w} - \tilde{w}}{\left[\sum (w_i - w)^2/N(N-1)\right]^{1/2}}.$$

The last expression is sometimes called *Bessel's Formula.*

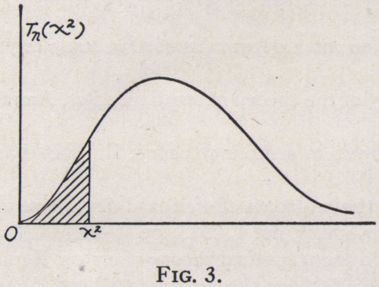


FIG. 3.

(c) *For the variance.* To determine the fiducial limits of σ^2 we first observe from (22a) that $Ns^2 = n\dot{\sigma}^2 = \Sigma(x_i - \bar{x})^2$, and therefore we may write $\chi^2 = n\dot{\sigma}^2/\sigma^2$. If now we make the claim

$$n\dot{\sigma}^2/\chi_2^2 < \sigma^2 < n\dot{\sigma}^2/\chi_1^2,$$

where χ_1^2 and χ_2^2 are arbitrarily chosen constants ($\chi_1^2 < \chi_2^2$), then our measure of confidence in the correctness of this claim is given by $I_n(\chi_2^2)-I_n(\chi_1^2)$, where (Figure 3)

$$I_n(\chi^2) = \int_0^{\chi^2} T_n(\chi^2)d\chi^2.$$

Tables of $I_n(\chi^2)$ are easily available [17].

Rietz [18] has explained this recent advance in statistical inference in a paper specially designed for teachers of collegiate courses in statistics. References are given there to the contributions in this connection of Fisher, Neyman, E. S. Pearson, Wilks, and others.

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JOSIAH WILLARD GIBBS*

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In the great glass of time the grains of sand are even now stirring which by their fall will soon mark the lapse of a century since the birth of Josiah Willard Gibbs, a man who must be placed at once among the greatest and the least generally known of American scientists.

To us, in this scientific group, it is a matter as we will it—the unexalted choice of a perspective—whether the century be regarded as a unit great or small. The geologist, the astronomer, the biological evolutionist, thinks effortlessly in eons by comparison with which the century is an insensible instant, and the scholar of the arts, the science, the philosophy or the mathematics of antiquity reaches aptly back over centuries, be they fifty or more. And yet, as students of modern science we remember that only three centuries or less have rolled by since Galileo paid in discouragement and anxiety the penalty of his heroic thought; since Descartes preached his scientific doctrine and founded modern mathematics; and since the mechanics of the world of today took shape in the great mind of Newton. The life of our national identity is not yet to be measured by two centuries, and even a single century ago the soil we tread was still in all truth that of a new land.

The American scientist of a hundred years ago stood forth, therefore, very much in the rôle of the pioneer. He won out in his achievements over many handicaps; without traditions at his back, with little in the way of schools and libraries or laboratories to his hand, and often in the face of small sympathy or encouragement from his neighbors, who were for the most part engrossed in the busy incidents peculiar to the industrial and commercial and agricultural enterprise of the new nation. The issues which were soon to lead to the test whether the bonds uniting the nation were tender or tough, were already inflaming the passions of men, and were all too well designed to overshadow the inconspicuous scholarly searchings and deductions, in the tiny impulses of which the revolutions of scientific discovery are generally born.

I need not recall to you the pomp and circumstance of a few years ago by which a great American city saw fit to mark the consummation of a century of scientific progress. There were exhibits to swell with pride the hearts of the disciples of science, and there was much that called to all mankind for homage on behalf of those rare men, who, facing the questions which the facts of nature constantly pose to the human mind, were of that superior wisdom that they could see the principles behind the facts, and grasping these place into the hands of their fellows the machineries of modern science for the control and understanding of nature and the consequent amelioration and enrichment of life. Of these rare men of extraordinary intellect was Josiah Willard Gibbs. An ancient proverb notwithstanding, the world-wide scientific authority of today is agreed that in his own field he was one of those uncommon men who could see tomorrow's sun.

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* An address to the Minnesota section of the Mathematical Association of America, October 28, 1938; also given at Northwestern University on November 30, 1938.

Josiah Willard Gibbs was born on the eleventh day of February in the year 1839, in the city of New Haven, Connecticut. His family had made its homes upon American soil from almost the earliest of colonial days, and had been identified over many generations with the finest in American thought and culture. Thus one finds among Gibbs's direct though more remote ancestors a President Willard of Harvard College, and a President Dickenson of the College of New Jersey which later became Princeton College, while Gibbs's father, also named Josiah Willard, was a professor at Yale, and an eminent scholar widely noted for the profundity of his thought in the fields of sacred literature and the classics. From men such as these Gibbs inherited genius and those traits of industry, power of concentration, and intellectual integrity, keenness, and modesty which were to characterize him throughout his life.

Though records of personal incidents or anecdotes of Gibbs's boyhood or youth are scant or utterly missing, it is not hard to reconstruct in essence the atmosphere of the environment in which he lived. New Haven was even then the seat of Yale College, and so was a center of American learning. From early colonial times it had been a community built largely around the church, and possessed of a strongly local and very individual character. Traditionally its code of preference was inclined in all matters toward a general and extreme conservatism, and this attitude, especially in matters of politics and religion, had been unwaveringly maintained. Although as a seaport it had cherished, and in those earlier days of the steamboat perhaps still did cherish, some dreams of commercial greatness, the life and ambitions of the city, as they expressed themselves through its more influential citizens, were focused in large measure upon the College. Yale was the acknowledged center of civic pride and ambition. It requires little in the way of imagination to conceive of how Gibbs, born into the family of a scholar and professor in such surroundings, must during his boyhood have been saturated with the atmosphere of collegiate life, and how he must have had instilled into him a thorough sense of the prime importance of things academic.

During these years the community of New Haven merited with far more justice a description as an overgrown village than as a city. Its connection with New York by railroad was not materialized until the year 1848, and in that same year street lighting by gas lamps was first introduced upon the main streets of the town. Until 1854 there were no graded schools, and for years beyond that fire protection was on a volunteer basis, and policing in the hands of a Department of the Watch. By common custom scavenger duty was delegated to swine and was allocated to the gutters of the city streets, and not until 1861, when Gibbs had already attained his majority, were pig-pens banished from the city, and a law passed abating the nuisance of horses and cows pasturing in the streets. On the main thoroughfares sidewalk paving was an innovation which had been bitterly opposed, and it is recorded that a prominent citizen still strode over this object of extravagance only when necessity compelled him to, and then in haste, and walked by preference in the street, fortified in his conviction that God's soil was still good enough for him.

Gibbs prepared himself for college at the Hopkins Grammar School in his home city. He was a distinguished student who divided his interests between mathematics and the classics. At the age of fifteen he entered Yale, and continued there to live up to his earlier promise. His interests were exclusively cultural, and so following his graduation he remained at Yale for five years more to receive his doctorate at the end of that time, in 1863. Yale had established the degree of doctor of philosophy only three years earlier, and in doing so had taken the lead among American institutions. With his degree Gibbs received an appointment as tutor in the College, and was assigned to teach classes at first in Latin, and then in Mathematics and Science. He found this assignment no easy one, nor one for which he was temperamentally well equipped, or in which expectations of brilliant success could be his. Whatever his genius, it was not that of the college teacher.

In physical aspect Yale presented at that time the picture of a row of old and homely buildings of brick, which were flanked round about by a campus noted for the distinction of its archways of great and stately elms. Along the street this campus was edged by a long rail fence around which much of Yale tradition entwined itself, and which from immemorial time had been the favorite roost of the students. As a barrier to ingress the fence was a negligible matter, and the campus, therefore, assumed much the character of a village green, where tramps, and beggars, and pedlars, and organ-grinders, and in fact all who would, mixed freely in the college life. As at noon the elms lent shade, they at night increased the darkness, and so unwittingly conspired to enhance the lure and convenience of the unlighted campus for all such as were bent upon purposes of academic mischief. The fugitive student could depend here at once upon ready access to the city streets and upon the asylum of many handy refuges and concealments, temptations which, it has been said, rarely failed to convert original saints into undergraduate sinners.

To the common creature comforts of the students the dormitories of the old brick row afforded only the rudest of satisfaction, and of this sort also were the class-rooms. The beams sagged, the floors were billowy and the ceilings cracked; and the walls were gouged and furrowed, for it had long been the prime ambition of every student to leave some lasting mark of his upon pillar or doorway. The rooms were illy lighted and seemed, therefore, steeped in permanent gloom. There was a general air of mustiness about and an appearance of roughness, and the sanitation was far from modern. Such details, however, seemed of small effect toward mitigating the intensity of life for the five hundred students of the College, and faculty vigilance enjoyed but rare respite from its appointed task of keeping pace with the students' inventive genius for pranks and insubordination.

With all this it was an academic era noted as one in which professors were, or at least were popularly regarded to be, highly individualized beings. Among them oddities of personality or habit were not regarded the exception but the rule, and even the younger tutors and teachers generally shared in these distinctions. All too commonly custom and their own expanding dignity placed them under the incumbency of assuming an aspect of austerity which made them appear as nothing so much as arch-foes to their charges. It was their first duty to match their discipline vigilantly against their opponent's ever-present urge for fun, and a tutor in actual physical pursuit of a fleeing undergraduate was by no means an uncommon sight.

For this sort of thing Gibbs could hardly have been more poorly endowed by nature. He was shy and retiring, gentlemanly, contemplative and reserved. The subjects he taught, moreover, Latin and Mathematics, were at that time required of all students throughout their first and second years, and, as is the way with requirements generally, these bred neither a desire to spare the tutor, nor a fondness for the subjects. Quite the contrary we may infer, for of the many campus customs at Yale in those times none appears to have been entered into with so much gusto and zest as the annual farcical pageant of the Burial of Euclid, with which the sophomore class was wont to celebrate its mathematical emancipation.

There are many records of this ceremonial in the Yale archives, and though in its details it naturally varied with the genius of the class, it maintained its identity in form over a period of generations. The sophomore class having been summoned to gloat over Euclid's death, assembled in some college hall which was bedecked suitably to the occasion. The scene was dominated by a large and lurid cartoon which bristled in detail with fire and fury, and depicted how in the presence of Jupiter demon stokers were assisting at the consumption of Euclid's remains in a sea of blazing tar. A dismal forest with embattled demons filled the remoter parts of the scene, while in the foreground a student visibly filled with despair lent company to a weeping crocodile. Under this aspect Euclid's volume was perforated with a glowing poker, each man of the class thrusting the iron through in turn to signify that he had gone through Euclid. Following this the book was held for a moment over each man to betoken that he had understood Euclid, and finally each man passed the pages under foot that he might say thereafter that he had gone over Euclid.

These preliminaries accomplished, the funeral cortege was formed, and proceeded lugubriously, with grotesque garb and blazing torchlights to the chosen place of interment. At times Euclid himself was impersonated, dressed in classic raiment and pressing his beloved volume to his breast, and at others the book alone was borne suitably shrouded at the head of the procession. At the pyre the celebration waxed in boisterousness and assumed more the aspects of revelry. There was elaborate mock lamentation, a funeral oration was held, and dirges more or less derisive were sung.

"No more we gaze upon that board
Where oft our knowledge failed,

As we its mystic lines ignored,
On cruel points impaled."

* * * *

"We're free! Hurrah! We've got him fast
Old Euk is nicely caged at last."

* * * *

"Black curls the smoke above the pile
And snaps the crackling fire:
The joyful shouts of Merry Sophs
With wails and groans conspire.
May yells more fiendish greet thy ears,
And flames yet hotter glow;
May fiercer torments rack thy soul
In Pluto's realms below."

As a student at Yale, Gibbs must have participated in such a rite, and during his life he must have witnessed it many times. For him personally, however, Euclid never died in any but a metaphoric sense; for geometry and geometric imagination were the corner-stone and buttress of his genius.

At the expiration of his appointment as tutor Gibbs went to Europe for further study. The Civil War, which had been running its course, ravaging the country and depressing personal incentive, had meanwhile come to its close, and in departing this country Gibbs figuratively became one of a brilliant troop of young intellectuals who were destined to play a decisive rôle in the cultural development of America. These men went to Europe in search of learning, and they found the fulfillment of their quest at the great German universities. There they found great minds in numbers and under conditions which made those minds accessible to others less mature. They found there also a breadth of academic viewpoint, a freedom of research, and an insistence upon productive scholarship which they had not theretofore known. Upon their return they brought back with them these ideals and perspectives, together with an abundance of enthusiasm; and under the spur of this inspiration they became the institutors of the system of post-graduate instruction and research which is the essence of the modern American university, and became the founders of the many learned societies which are today conspicuous forces in the intellectual and cultural life of the nation.

Gibbs spent three years in Europe, studying for a time at Paris, but principally at Heidelberg, as a student of those great teachers, Helmholtz and Kirchhoff, and at Berlin under the influence of the supreme rigorist genius of the great Weierstrass. Subsequent to his return to America, Gibbs was elected, in 1871, to the Professorship of Mathematical Physics at Yale. He was then thirty-two years old, and was to hold this professorial chair without interruption for precisely that many years again until his death. The distinction which had thus been bestowed upon him was apparently a cheap one for Yale, and one which for Gibbs remained long empty of all but honor. During many years it carried no remuneration at all, and during many more only a fragment of an otherwise customary salary.

Had Gibbs been under the necessity of maintaining himself, his position at Yale would, of course, have been untenable. Fortunately this was not so. His father had bequeathed him a competence which, though small, was matched by the smallness of his need. He was, and always remained unmarried. Throughout his life he retained the occupancy of rooms in the old family house in which he had spent all the boyhood he could remember, and which stood in close proximity to the College campus. This house sheltered now the family of a married sister, and in this family he found his own permanent home. His life was routine and uneventful; for social contacts he felt but little need, and the craving to see and hear, which impels one to travel, was not his. He had few aesthetic needs, was abstemious in his habits, and seemed to find in his work all that he sought from life.

The study which claimed the initial interest of Gibbs as a professor was that of thermodynamics, the science which treats of heat as a form of energy, and concerns itself with the laws governing the transformations of heat into energy in different forms. This was then a new science, and for Gibbs it was an absorbing one. The results of his first two years of research were given out by him in the form of two papers which were respectively entitled "Graphical Methods in the Thermodynamics of Fluids," and "A Method of Geometrical Representation of the Thermodynamic Properties of Substances by Means of Surfaces." The papers were remarkable. By using as mathematical coördinates such physical quantities as volume and pressure, energy, temperature and entropy, they derived, on the one hand, heat diagrams of various types which later became instruments of great importance for the thermal engineer, and, on the other hand, showed how with any physical body there might be associated a graph or so-called thermodynamical surface, from the geometrical configuration of which could be recognized the many relations between volume, energy, temperature, pressure and entropy, the conditions for stability and equilibrium, and the passage from the liquid to the solid or gaseous states. In these papers Gibbs revealed himself at once as possessed of a rare imagination in the domain of abstract geometry, and as a master in its application.

These initial works of Gibbs's genius were followed in 1876 and 1878, that is, in his thirty-seventh and thirty-ninth years, by his greatest memoir, "On the Equilibrium of Heterogeneous Substances." Here Gibbs rose to the pinnacle, and revealed himself to be a true intellectual giant. The work is a monumental one, immense in its scope, and one which shows, as almost no other scientific work does, the sheer power of human thinking. In the science of physics the law of the conservation of energy, under the transformation of mechanical work into heat and vice versa, was even in Gibbs's time a familiar instrument, and one of the most effective, in the hands of the theoretical investigator, to his purpose of deducing from observed phenomena an intelligible picture of nature. In this respect the science of chemistry was far behind, for the relations between the energies of chemical reactions and heat had almost completely eluded all attempts to bring them within the realms of scientific law. It was to the difficulties of this problem that Gibbs had bent his thought, and these difficulties he had at one stroke subjugated with an astounding completeness.

The paper stands as a great model of the rôle which mathematics rightly plays in its relations with the sciences. Assuming command over a bewildering welter of apparently unrelated facts, it imposes upon them a few fundamental laws, and reduces the whole to rule and order. Gibbs based his authority upon the first two laws of thermodynamics, namely, the law of the conservation of energy and the law that heat will not of itself flow from a colder body to a hotter one. To these laws he adjoined a few experimentally determined primary chemical facts, and from this basis proceeded by mathematical deduction alone, with unbending rigor, to clear his intellectual way step by step, and to uncover again and again a basic principle and intrinsic likenesses between things in which such had superficially seemed remote. Specifically, Gibbs deals in this great work with the statics of chemical substances which are in contact with each other, and derives for them conditions for their co-existence, their equilibrium, or their stability as solids, liquids, vapors, or gases, or as liquid films, gaseous mixtures, solutions, or crystals, and discusses the effects upon them of osmosis or gravity, of electromotive or capillary or catalytic forces. In its reasoning it is a true unfolding of nature's law, and in its results it laid the foundation of a new science, a science of great present-day vitality, the science of Physical Chemistry.

Gibbs's great achievement was slow to attain its deserved and destined influence, and for this there were many reasons. As a scientist he was a thoroughly solitary figure, and the researches of his paper constituted a scientific departure which in its originality had been entirely unforeshadowed by the work of others. He had had no helpers, and as a true innovator he had no rivals. His paper was modestly published in the *Transactions of the Connecticut Academy of Arts and Sciences*, and was, therefore, in large measure obscure, and certainly in Europe almost inaccessible. Finally his paper was of a most forbidding aspect, as Gibbs was certainly no easy writer to read. His style was bare and concise, the cast of his ideas was severe, and his reasoning prompt, unerring and rigorous. Of emphasis he gave little, and a thing once said was done with. Finally, but critically, the great treasures he uncovered were treasures for the chemist, whereas his paper barely mentioned as many as five or six chemicals, all of them simple, and at the same time extended over three hundred printed pages which are covered with some seven hundred mathematical formulas. Such a memoir, one may venture, would be no mean test of mettle for the prospecting chemist of today. Small wonder that it remained largely unexplored some three score years ago.

The past half century has spoken in emphatic term for the brilliance and profundity of Gibbs's achievement. His paper is rated as a preeminent document of scientific prophecy, for many phenomena which it predicted, which were at the time unknown and unsuspected, have since been discovered or rediscovered by experimental means. The principles which Gibbs laid down have led to a wealth of original and fruitful researches which even now are apparently far from exhaustion. Emerson might well have said of them, "The creation of a thousand forests is in one acorn."

There is every reason to believe that Gibbs was himself fully aware of the great ultimate importance of his work. Nothing would have been more foreign to him, than to have lent word or action of his own to further its recognition or acclaim. It was a conspicuous characteristic of his nature to mantle his own personality with a distinctive cloak of reserve. In his personal contacts he seems to have been always friendly and considerate, kindly and affable and of a ready and spontaneous if somewhat subdued sense of humor. As an intellect, however, he withdrew into himself. Though his associates and colleagues easily recognized him as having in full measure those traits of character which Francis Bacon held characteristic of the true scholar, "the desire to seek, the patience to doubt, fondness to meditate, slowness to assert, readiness to reconsider, carefulness to dispose and set in order, and repugnance to every kind of imposture", they were permitted to recognize beyond this none of the more intimate things which filled his mind. He loved his work, and was possessed of an impelling enthusiasm for it, but these matters he kept in concealment. He worked without either the stimulus of conversation or that of criticism, and never spoke of his ideas until they were rounded out and in every way ready for publication.

Herein undoubtedly lay Gibbs's greatest failing. By profession he was a teacher, by temperament he lacked entirely the teaching spirit. Silent and reserved men have often been great as teachers, but this was not so with Gibbs. He taught only graduate students, and his students were never more than a few, but even to these he never confided the matters which at any time were at the focal point of his interest. To the few who had the will and the ability to follow him he was inspiring. All found his lectures difficult. They were invariably well prepared, but their thought was heavily concentrated, the progression of ideas often precipitate, and exercises, if such were included at all, were all too often and readily brushed aside.

The repertoire of courses from which Gibbs lectured from year to year seems to have been essentially the following: Vector Analysis, Capillarity, The Wave Theory of Light and Sound, Least Squares, The Theory of Potential, The Mathematical Theory of Electricity and Magnetism, and Multiple Algebra, and later,—not until fifteen years after his great memoir,—Thermodynamics, Statistical Mechanics, and The Computation of Orbits.

Much of the theory set forth in these lectures was original. This was so, for instance, with the Vector Analysis, a creation of Gibbs's for which almost all scientists of the present day are indebted to him. The Cartesian coördinate geometry, powerful mathematical tool though it be, becomes involved and unwieldy in the extreme when it is applied to elaborate space relations, or to the study of strains, twists, spins, or other common aspects of rotational motion. The desirability of an emancipation from it had been recognized by many mathematicians before Gibbs, and had led the German mathematician Grassmann to his "Ausdehnungslehre," and the Englishman Hamilton to his Theory of Quaternions. This last as a theory is concise, consistent and elegant. As an instrument, however,—and though it had been used with masterly effect by Maxwell,—it had generally been found unfortunately artificial. The mathematician Cayley speaking to this point, compared Quaternions to a pocket-map, which to be used has to be unfolded. The quaternion formula, he felt, had to be retranslated into coördinates to be really understood.

These defects Gibbs sought to overcome through the medium of a sort of fusion of the German and English works. The result was his creation of Vector Analysis, which during the lapse of half a century has established itself as an indispensable tool of the theoretical scientist. With customary modesty Gibbs refused to regard his work here as properly original. He said of it: "The notions are only those which he who reads between the lines will meet on every page of the great masters of analysis, the only difference being that the vector analyst, having regard for the weakness of the human intellect, does as the early painters did who wrote beneath their pictures, 'This is a tree and this a horse'." He had an account of the Vectors printed privately for the use of his students in 1881. Only in 1901 did he consent to the publication of a book on the subject.

During the years 1882 to 1889 Gibbs published a series of papers on the theory of light, and an important paper for the astronomer, "On the Determination of Elliptic Orbits from Three Observations." Of these the latter has become classical, and has achieved an immense saving in astronomical calculations. The former constitute together what is generally regarded as the simplest and most conclusive argument on purely theoretical grounds for the acceptance of the electromagnetic theory of light. At this day the correctness of his contentions has, of course, been long established, by those experimental results which meant the triumph of Maxwell's theory.

The genius of Gibbs had meanwhile received recognition, not alone in this country but throughout the world. Though to the layman, and even to many of his immediate colleagues in other fields, he remained unknown, and though he has been called an author whose books no one of his generation was ready to read, he had been elected to membership in the National Academy of Sciences, and had been awarded the Rumford Medal of the American Academy of Arts and Sciences at Boston. Before his death he was to hold honorary degrees from universities in three different countries, and to become a corresponding member of fifteen of the world's great learned societies. He was to receive from the Royal Society of London its Copley Medal, the highest distinction for research in any land, and to be elected to a vice-presidency of the American Association for the Advancement of Science.

In his final work, "The Principles of Statistical Mechanics," Gibbs was again to step forth as the innovator, and to open a most fertile field for subsequent extended scientific investigation. The belief had long been common among scientists that heat in a substance was due to motion of the molecules, but simple as this conception was, it had persistently defied all efforts aimed toward its theoretical demonstration. This failure was stigmatized by the Physicist Kelvin as "a cloud upon the history of science in the nineteenth century." To remedy it Gibbs undertook the study of mechanical systems which are composed of vast aggregates of particles. Though these particles individually were to be regarded as obeying the classical Newtonian laws of motion, a consideration of them, in the face of their great number, would have been hopeless. He took, therefore, as cornerstones upon which to build, the laws of averages and of probability, and with none but the simplest of mechanical assumptions as tools proceeded to erect his theory. From it emerged in fine succession all the basic laws of heat as they are embodied in the science of thermodynamics.

Among American men of science Gibbs holds a preeminently high place. His was one of those rare intellects from which the race obtains its pictures of the world as a cosmic universe. His mind was one in which disjointed phenomena were organized, and such generalized statements of scientific law as mark epochs in the advance of exact knowledge were thought out. Such minds are very rare, and their thoughts are incalculable treasures. The mathematician finds peculiar satisfaction in the work of Gibbs, for in it is revealed the quintessential power of mathematics for "spreading its net over the Cosmos and calling forth from it order, abstract form, and the law of science."

Gibbs died suddenly in 1903, in the sixty-fourth year of his life. He left little in the way of notes, for he had always been accustomed to carry his unfinished work only in his head. It was known that he planned an extension of his great work in thermodynamics, but his thought in that connection will never be known. He waited with them too long.

"They do not die who leave their thought
Imprinted on some deathless page,
Themselves may pass: the spell they wrought
Endures on Earth from age to age."

THE DELTOID REGARDED AS THE ENVELOPE OF SIMSON LINES

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1. Introduction. Recently C. E. Van Horn (this MONTHLY, 1938, p. 434) has called attention to the envelope of the Simson lines of a triangle and proved by an ingenious use of coördinates the theorem that it is a hypocycloid of three cusps, or deltoid, circumscribed about the ninepoint circle. This and his other results, the location of the cusps and the points of contact on the sides and altitudes of the given triangle, will be treated in this paper by synthetic methods. The other questions considered in this paper concern the construction of tangents to the deltoid from points of the inscribed circle, the equal but oppositely directed rotation of tangents whose intersection moves along a given tangent, and the fact that the lengths of two tangents are in the same ratio as the sines of the angles which they make with the third tangent through their intersection. The purely geometrical method for treating the envelope and constructing the tangents from points of the ninepoint circle seems not to be generally known.*

2. The Envelope. To identify the envelope of the Simson lines we need only note that each such line passes through a point P' of the ninepoint circle midway between the orthocenter H and its pole P, and that if P and P' describe arcs determined by central angles 2θ, the Simson line rotates through the angle $-\theta$. (Johnson, p. 206.) But this is exactly the behavior of the tangent to the deltoid. To verify this we may note that if the point of contact C of the rolling circle (O') with the fixed circle (O) is determined by the parametric angle 2θ, the angle $CO'D$, then D is the point of the deltoid, is 6θ. The line $P'D$ is tangent to the curve, where P' is the point of contact of the rolling circle with the inscribed circle. The point D is the foot of the perpendicular dropped from C on the tangent, an observation which is useful in proving Van Horn's theorems concerning the contacts of the sides and altitudes. To sketch the proof, it will be sufficient to note further that for an altitude the point P' lies midway between the orthocenter and the corresponding vertex, and for a side it is the midpoint of that side. To conclude the present remarks about the rotation of the tangent to a deltoid, we note that the angle $CP'D$ is 3θ or that the line $P'D$ has rotated through the angle $-\theta$ while OP' rotates through the angle 2θ.

To locate the cusps, we note that a cusp tangent passes through the ninepoint center. Consider the behavior of the Simson line as its support P' moves from the position M, midway between H and a vertex, towards the point K at the extremity of the radius of the ninepoint circle parallel to, and in the same direction as, HM. If K also moves towards P' but only half as fast as P', the Simson line and the radius will remain parallel and will coincide when P' has come to a position two-thirds of the way from M to the initial position of K.

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* See R. A. Johnson, Modern Geometry, 1929, p. 212.

3. Construction of Tangents. The books by Altshiller-Court* and Johnson both give methods for constructing the Simson line parallel to a given line. After the cusp tangents have been drawn, we may give another very simple construction using again the relation that the Simson line which makes the angle $-\theta$ with a cusp tangent NX passes through a point P' of (N) such that the angle XNP' is 2θ. In addition to the points at infinity from which tangents can be constructed, as noted by Johnson, we can construct the tangents from P' of (N) using the angles between NP' and the three cusp tangents in turn.

The problem of finding the tangents from a general point to the deltoid can be reduced to finding the points of intersection of the ninepoint circle and a certain cubic having three asymptotes parallel to the cusp tangents of the deltoid. This curve is determined as the intersection of NP' and a line through the given point parallel to the Simson line through P'. The portion of the cubic generated by the positive ray NP' cuts the ninepoint circle in one or three points.

4. Theorems Concerning Tangents. It is known† that the orthopole of a line cutting the circumcircle at P and Q is the intersection of the Simson lines of P and Q, and the locus of the orthopole as the line moves parallel to itself is another Simson line. Since P and Q move over equal but oppositely directed arcs, we can announce for the deltoid the following theorem:

THEOREM. *If two tangents to a deltoid rotate through equal but oppositely directed angles, their intersection describes a third tangent, and conversely.*

This result can be applied very simply to obtain another. Consider the two triangles formed respectively by the initial and terminal positions of the two moving tangents and having the side along the fixed tangent in common, whose length we shall call a. Let the equal angles at A and A' opposite the side a be α, and let the initial and terminal points of a be called B and C respectively. Let y be the angle ACB and y' the angle $A'CB$. Then

$$CA/\sin(\alpha + y) = a/\sin\alpha = CA'/\sin(\alpha + y').$$

Let C move back into coincidence with B, whereupon α approaches zero and A and A' approach the points of contact on the tangents BA and BA'. Thus we arrive at the following theorem:

THEOREM. *The segments from the point of intersection of two tangents to a deltoid to the respective points of contact are proportional to the sines of the angles which the two tangents make with the third tangent through the point.*

---

* College Geometry, 1925.

† Johnson, p. 247.

AN ACCURATE METHOD FOR OBTAINING THE DERIVATIVE FUNCTION FROM OBSERVATIONAL DATA

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1. Introduction. In the case of observational data proceeding at constant increments of the independent variable, Rutledge [1] has given a method for the determination of the derivative function. The formulas which he gave may also be obtained as follows.

We fix attention upon the values of y (the dependent variable) corresponding to $x=x_0, x_1, \cdots, x_8$ (where Δx is constant) and assume a fourth degree polynomial approximation for $y=f(x)$ in the interval $x_0 \leqq x \leqq x_4$. Newton's interpolation formula [2] gives

$$y = y_0 + \binom{u}{1}\Delta y_0 + \binom{u}{2}\Delta^2 y_0 + \binom{u}{3}\Delta^3 y_0 + \binom{u}{4}\Delta^4 y_0, \tag{1}$$

where

$$\binom{u}{j} = \frac{u(u-1)\cdots(u-j+1)}{j!}, \quad \text{and where } x = x_0 + u\cdot\Delta x.$$

We differentiate with respect to u and evaluate at $u=0, 1, 2, 3, 4$. In a similar manner we apply the corresponding formula for the five points starting with x_1, then with x_2, then with x_3, and finally with x_4.

We thus obtain five determinations for dy/du at $u=4$ as follows:

$$\begin{aligned}
z_0 &= \Delta y_0 + \frac{7}{2}\Delta^2 y_0 + \frac{13}{3}\Delta^3 y_0 + \frac{25}{12}\Delta^4 y_0, \\
z_1 &= \Delta y_1 + \frac{5}{2}\Delta^2 y_1 + \frac{11}{6}\Delta^3 y_1 + \frac{1}{4}\Delta^4 y_1, \\
z_2 &= \Delta y_2 + \frac{3}{2}\Delta^2 y_2 + \frac{1}{3}\Delta^3 y_2 - \frac{1}{12}\Delta^4 y_2, \\
z_3 &= \Delta y_3 + \frac{1}{2}\Delta^2 y_3 - \frac{1}{6}\Delta^3 y_3 + \frac{1}{12}\Delta^4 y_3, \\
z_4 &= \Delta y_4 - \frac{1}{2}\Delta^2 y_4 + \frac{1}{3}\Delta^3 y_4 - \frac{1}{4}\Delta^4 y_4.
\end{aligned} \tag{2}$$

We make use of familiar relations such as $\Delta y_0 = y_1 - y_0$, $\Delta^2 y_0 = y_2 - 2y_1 + y_0$, to obtain:

$$\begin{aligned}
12z_0 &= 3y_0 - 16y_1 + 36y_2 - 48y_3 + 25y_4, \\
12z_1 &= -y_1 + 6y_2 - 18y_3 + 10y_4 + 3y_5, \\
12z_2 &= y_2 - 8y_3 + 8y_5 - y_6, \\
12z_3 &= -3y_3 - 10y_4 + 18y_5 - 6y_6 + y_7, \\
12z_4 &= -25y_4 + 48y_5 - 36y_6 + 16y_7 - 3y_8.
\end{aligned} \tag{3}$$

Either set of formulas may be used although the second set is much easier to use if a computing machine is available.

Similar sets of formulas could be obtained by the use of "sliding" curves of other degrees than the fourth.

2. The weighted average. In formulas (1) we drop the hypothesis of a fourth degree polynomial approximation, continue the series, evaluate the derivatives, and obtain as error series for the formulas (2) the following:

$$
\begin{aligned}
R_0 &= \frac{1}{5}\Delta^5 y_0 - \frac{1}{30}\Delta^6 y_0 + \frac{1}{105}\Delta^7 y_0 - \frac{1}{280}\Delta^8 y_0 + \cdots, \\
R_1 &= -\frac{1}{20}\Delta^5 y_1 + \frac{1}{60}\Delta^6 y_1 - \frac{1}{140}\Delta^7 y_1 + \frac{1}{280}\Delta^8 y_1 - \cdots, \\
R_2 &= \frac{1}{30}\Delta^5 y_2 - \frac{1}{60}\Delta^6 y_2 + \frac{1}{105}\Delta^7 y_2 - \frac{1}{168}\Delta^8 y_2 + \cdots, \\
R_3 &= -\frac{1}{20}\Delta^5 y_3 + \frac{1}{30}\Delta^6 y_3 - \frac{1}{42}\Delta^7 y_3 + \frac{1}{56}\Delta^8 y_3 - \cdots, \\
R_4 &= \frac{1}{5}\Delta^5 y_4 - \frac{1}{6}\Delta^6 y_4 + \frac{1}{7}\Delta^7 y_4 - \frac{1}{8}\Delta^8 y_4 + \cdots.
\end{aligned} \tag{4}
$$

We form $\sum_{i=0}^{4} w_i(z_i+R_i)$ and then so choose these weight parameters $w_0, w_1, \cdots, w_4$ that, if differences of order nine are zero, $\sum_{i=0}^{4} w_i R_i = 0$. An easy computation yields four homogeneous equations in these weight parameters, and the general solution is $w_0=w_4=k$, $w_1=w_3=16k$, $w_2=36k$. *Hence a weighted average of the five determinations from formulas* (2) *or* (3), *using weights* 1, 16, 36, 16, 1 *respectively, yields a value for* dy/du *at* $u=4$ *which is exact if* $y=f(x)$ *is a polynomial of degree eight or less.*

If the middle three determinations are utilized and if we assume differences above the sixth order to be zero, a similar procedure shows the proper weights to be 1, 3, 1. *This resulting weighted average is precise if* $y=f(x)$ *is a polynomial of degree six or less.*

Let the former weighted average be W_5 and the latter be W_3. Then

$$
\begin{aligned}
W_5 &= (z_0 + 16z_1 + 36z_2 + 16z_3 + z_4)/70, \\
W_3 &= (z_1 + 3z_2 + z_3)/5.
\end{aligned} \tag{5}
$$

We substitute from equations (3) to obtain:

$$
\begin{aligned}
W_5 &= (3y_0 - 32y_1 + 168y_2 - 672y_3 + 672y_5 - 168y_6 + 32y_7 - 3y_8)/840, \\
W_3 &= (-y_1 + 9y_2 - 45y_3 + 45y_5 - 9y_6 + y_7)/60.^*
\end{aligned} \tag{6}
$$

---

* *Note*: Either of these two formulas could be used for calculating dy/du at $u=4$ directly from the data. The advantage in the former method lies in the fact that one can then see how much of the final result is reasonably certain, *i.e.*, common to the several estimates [**1**].

Also W_5 is the formula for dy/du at $u=4$ that one would obtain by assuming an eighth degree polynomial approximation for $y=f(x)$ which passes through the points with abscissas $x_0, x_1, \cdots, x_8$, and by evaluating the derivative of this polynomial at $u=4$. A similar statement holds for W_3.

3. Estimates for errors. The errors in the estimates z_i and in the weighted averages W_3 and W_5 will be due to two sources: errors in the original data, errors due to method.

Let $E(z)$ denote the maximum possible error in z due to a maximum possible error of k in each of the original values of y. Then one can easily find, using formulas (3) and (6):

$$
\begin{aligned}
&E(z_0) = E(z_4) = \frac{32}{3}k, \qquad E(z_1) = E(z_3) = \frac{19}{6}k, \qquad E(z_2) = \frac{3}{2}k,\\
(7)\qquad &E(W_5) = \frac{25}{12}k, \qquad E(W_3) = \frac{11}{6}k.
\end{aligned}
$$

Let q denote the probable error of each entry y in the original data. Let $H(z)$ denote the probable error of z. We use formulas (3) and (6) to obtain:

$$
\begin{aligned}
&H(z_0) = H(z_4) = \frac{1}{12}q(3^2 + 16^2 + 36^2 + 48^2 + 25^2)^{1/2} \approx 5.58q;^*\\
(8)\qquad &H(z_1) = H(z_3) \approx 1.81q; \qquad H(z_2) \approx 0.95q;\\
&H(W_5) \approx 1.17q; \qquad H(W_3) \approx 1.08q.
\end{aligned}
$$

Estimates for the error, due to method, in each z_i are given by the first terms of the series in (4). We substitute these first terms in the first equation in (5) to obtain an estimate for the error, due to method, in W_5, namely $G(W_5)$:

$$
\begin{aligned}
G(W_5) &= \frac{1}{350}\Delta^5(y_0 - 4y_1 + 6y_2 - 4y_3 + y_4),\\
(9)\qquad &= \frac{1}{350}\Delta^5(1 - E)^4 y_0, \quad \text{where} \quad Ey_0 = y_1, \quad etc.,\\
&= \frac{1}{350}\Delta^9 y_0, \quad \text{since} \quad \Delta = E - 1 \ [3].
\end{aligned}
$$

Similarly,

$$
(10)\qquad G(W_3) = -\frac{1}{100}\Delta^7 y_1.
$$

We compare the results of formulas (8), (9), (10) and observe that on the basis of errors in the original data, z_2 is a better estimate than z_0, z_1, z_3, z_4, W_3, or W_5. On the basis of errors due to method, W_5 is the best estimate and W_3 is next. Since

$$
\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = \frac{1}{\Delta x}\,\frac{dy}{du},
$$

the errors in dy/dx are $1/\Delta x$ times the errors in dy/du. Hence in a particular problem, Δx should be large enough to make the probable error negligible and small enough to make the error due to method negligible.

---

* The symbol $\approx$ is used to mean "approximately equal."

Since the computation of W_3 involves the use of only the middle three determinations, there is a saving of labor in the use of W_3 in place of W_5.

4. A numerical example. To illustrate the method, we utilize the same illustration as given by Rutledge, namely, the data in Table I obtained by rounding off a twenty-three place table to ten places of decimals [4].

TABLE I

x	$\sin x$	x	$\sin x$	x	$\sin x$
0.6	0.56464 24734	0.9	0.78332 69096	1.2	0.93203 90860
0.7	0.66421 76872	1.0	0.84147 09848	1.3	0.96355 81854
0.8	0.71735 60909	1.1	0.89120 73601	1.4	0.98544 97300

We apply formulas (3) and (5) to obtain (at $x=1$):

$$
\begin{array}{llll}
z_0 = 0.05402 & 88880 & 5, & \qquad z_4 = 0.05402 \quad 94454 \quad 3, \\
z_1 = 0.05403 & 05344 & 7, & \qquad W_3 = 0.05403 \quad 02302 \quad 5, \\
z_2 = 0.05403 & 00507 & 4, & \qquad W_5 = 0.05403 \quad 02306 \quad 3. \\
z_3 = 0.05403 & 04645 & 4, &
\end{array}
$$

Since $\Delta x=0.1$, the value of dy/dx is obtained by multiplying each of these estimates by 10. The correct value is 0.54030 23059.

Table II is a section of a larger table. The second column gives the value of $\sin x$ correct to the nearest fifth decimal. The third, fourth, and fifth columns give the estimates z_1, z_2, z_3, each divided by Δx. The sixth column gives $W_3/\Delta x$. The seventh column gives the values of $dy/dx=\cos x$, correct to the nearest fifth decimal.

TABLE II

x	$\sin x$	z_1	z_2	z_3	W_3	$\cos x$
0.5	0.47943	0.87759	0.87755	0.87750	0.87755	0.87758
0.6	0.56464	0.82527	0.82532	0.82537	0.82532	0.82534
0.7	0.64422	0.76493	0.76488	0.76488	0.76489	0.76484
0.8	0.71736	0.69670	0.69671	0.69672	0.69671	0.69671

In this example, the probable error q of each entry is 0.000,002,5. We use the formula

$$\Delta^n(\sin x) = \left(2 \sin \frac{\Delta x}{2}\right)^n \sin\left(x + n\,\frac{\Delta x + \pi}{2}\right)$$

and easy computations to show that $|G(z_1)| < 0.000{,}000{,}38$; $|G(z_2)| < 0.000{,}000{,}25$; $|G(z_3)| < 0.000{,}000{,}38$; $|G(W_3)| < 0.000{,}000{,}000{,}7$; $H(z_1) = H(z_3) \approx 0.000{,}004{,}5$; $H(z_2) \approx 0.000{,}002{,}4$; $H(W_3) \approx 0.000{,}002{,}7$. To obtain estimates for the errors in dy/dx, each of these numbers must be multiplied by $1/\Delta x = 10$. Hence the final accuracy of $10W_3$ is about one-tenth that of $y=\sin x$, since $H(10W_3) \approx 10H(y)$ where $H(10W_3) \approx 0.000{,}027$ and $H(y) = 0.000{,}002{,}5$.

5. A general weight theorem. The two sets of weights previously obtained form two illustrations of the general theorem:

THEOREM: *Let the values of* $y=f(x)$ *be given for* $x=x_0, x_1, \cdots$, *where* Δx *is constant. Compute* $n+1$ *determinations for* dy/du *at* $u=n$ *(where* $x=x_0+u\Delta x$*) by the use of a "sliding" n-ic and let these be* $z_0, z_1, \cdots, z_n$ *respectively. Form a weighted average of these* z_k, *using as weights the coefficients in*

$$\frac{\partial^{p+q}}{\partial\alpha^p\partial\beta^q}\sum_{j=0}^{n}\binom{n}{j}^2\alpha^{n-j}\beta^j, \qquad p+q<n,$$

(the weight for z_0 is the coefficient of α^n, and so on). *The weighted average will be the precise value of* dy/du *at* $u=n$ *if* $y=f(x)$ *is a polynomial of degree* $2n-(p+q)$ *or less.*

Proof: The Newton interpolation formula, for $y=f(x)$ a polynomial of degree $2n$, is

$$y = \sum_{j=0}^{2n}\frac{u^{(j)}}{j!}\Delta^j y_0, \qquad x = x_0+u\Delta x, \tag{11}$$

$$u^{(j)} = u(u-1)(u-2)\cdots(u-j+1).$$

Then

$$\frac{dy}{du} = \left\{\sum_{j=0}^{n}+\sum_{j=n+1}^{2n}\right\}\left\{\frac{\Delta^j y_0}{j!}\frac{du^{(j)}}{du}\right\}.$$

By sliding indices on $\Delta^j y_0$, we obtain, for $k=0, 1, \cdots, n$,

$$z_k = \sum_{j=0}^{n}\frac{\Delta^j y_k}{j!}\frac{du^{(j)}}{du}\Big]_{u=n-k}, \tag{12}$$

and

$$\begin{aligned} R_k &= \sum_{j=n+1}^{2n}\frac{\Delta^j y_k}{j!}\frac{du^{(j)}}{du}\Big]_{u=n-k} \\ &= \sum_{j=n+1}^{2n}(-1)^{k+j-n-1}\frac{(k+j-n-1)!(n-k)!}{j!}\Delta^j y_k. \end{aligned} \tag{13}$$

Each of the values $z_k+R_k(k=0,1, \cdots, n)$ is a precise value for dy/du at $u=n$, if $y=f(x)$ is a polynomial of degree $2n$ or less.

Since differences of order higher than $2n$ are zero,

$$\Delta^j y_k = \sum_{l=j}^{p}\binom{k}{l-j}\Delta^l y_0, \tag{14}$$

where p is the smaller of $k+j$ and $2n$. We form

$$\psi(\alpha, \beta) = \sum_{k=0}^{n} \binom{n}{k}^2 R_k \alpha^{n-k}\beta^k. \tag{15}$$

The theorem will be established if we can show that

$$\frac{\partial^{p+q}\psi(1, 1)}{\partial\alpha^p\partial\beta^q} = 0, \qquad (p, q = 0, 1, \cdots, p + q < n).$$

We substitute equations (13) and (14) in (15) to obtain

$$\psi(\alpha, \beta) = \sum_{k=0}^{n} \sum_{j=n+1}^{2n} \sum_{l=j}^{p} \frac{(n!)^2(-1)^{k+j-n-1}(k + j - n - 1)!\alpha^{n-k}\beta^k\Delta^l y_0}{(l - j)!(k)!(n - k)!(k - l + j)!}.$$

We rearrange this finite summation and obtain:

$$\begin{aligned} \psi(\alpha, \beta) &= \sum_{l=n+1}^{2n} A_l(\alpha, \beta)\Delta^l y_0, \\ A_l(\alpha, \beta) &= \sum_{j=n+1}^{l} \sum_{k=l-j}^{n} \frac{(-1)^{k+j-n-1}(n!)^2(k + j - n - 1)!\alpha^{n-k}\beta^k}{(k)!(j)!(n - k)!(l - j)!(k - l + j!)}. \end{aligned} \tag{16}$$

If, for brevity, we set $\gamma = (\alpha - \beta)^n$, we can write this in the form

$$\begin{aligned} A_{n+m}(\alpha, \beta) = \frac{n!}{(n + m)!}\Bigg\{&\binom{n + m}{m - 1}\beta^{m-1}\frac{\partial^{m-1}\gamma}{\partial\beta^{m-1}} - \binom{n + m}{m - 2}\beta^{m-2}\frac{\partial^{m-1}(\beta\gamma)}{\partial\beta^{m-1}} \\ &+ \cdots + (-1)^{m-1}\binom{n + m}{0}\frac{\partial^{m-1}}{\partial\beta^{m-1}}(\beta^{m-1}\gamma)\Bigg\}. \end{aligned}$$

Then

$$\frac{\partial^{p+q}A_{n+m}(1, 1)}{\partial\alpha^p\partial\beta^q} = 0, \qquad \begin{aligned} &m = 1, 2, \cdots, n - p - q; \\ &p, q = 0, 1, 2, \cdots, n; p + q < n, \end{aligned}$$

since each term contains a partial derivative of $\gamma = (\alpha - \beta)^n$. If $y = f(x)$ is a polynomial of degree $2n - (p + q)$, the limits on the summation, in equation (16), will be from $n + 1$ to $2n - (p + q)$, since $\Delta^l y_0$ will be zero for $2n - p - q < l \leqq 2n$. Hence the theorem is established.

References

1. George Rutledge, Reliable method of obtaining the derivative function from smoothed data of observation, Physical Review, vol. 40, no. 2, 1932, pp. 262–268.

2. Whittaker and Robinson, Calculus of Observations, page 12.

3. Boole, Finite Differences, 1931, page 17.

4. C. E. Van Orstrand, Tables of the exponential function and of the circular sine and cosine to radian argument, Memoirs of the National Academy of Sciences, vol. 14, 1921, Fifth Memoir.

RECENT PUBLICATIONS

EDITED BY VIRGIL SNYDER, Cornell University

All books for review should be sent directly to the editor of this department, at the Mathematical Association of America, 531 West 116th Street, New York, N. Y., and not to any of the other editors or officers of the Association.

NEW BOOKS RECEIVED

Advanced Mathematics for Engineers. By H. W. Reddick and F. H. Miller. London, Chapman and Hall, Limited, 1938. 10+473 pages. $4.00.

Elementary Mathematical Statistics. By W. D. Baten. New York, John Wiley and Sons, Inc.; London, Chapman and Hall, Limited, 1938. 10+338 pages. $3.00.

Statistical Methods. By George W. Snedecor. Second Edition. Ames, Iowa, Collegiate Press, Inc., 1938. 13+388 pages. $3.75.

Band III, Astronomia Nova. By Johannes Kepler, Gesammelte Werke. (Herausgegeben von Max Casper.) München, C. H. Beck'sche Verlagsbuchhandlung, 1937. 488 pages.

Introductory Quantum Mechanics. By Vladimir Rojansky. New York, Prentice-Hall, Inc., 1938. 10+544 pages. $5.50.

A New Geometry. By A. W. Siddons and K. S. Snell. Cambridge, The University Press, 1938. 16+304 pages. $1.32.

Diophantische Gleichungen. By Th. Skolem. Ergebnisse der Mathematik und ihrer Grenzgebiete, 1938. 10+120 pages. RM. 15.

Introduction to Fourier Series and Boundary Value Problems. By Ruel V. Churchill. Ann Arbor, Michigan, Ruel V. Churchill, 1938. 4+94 pages. $2.50.

Leitfaden der Stereometrie. By W. Benz. (Mathematisches Unterrichtswerk für höhere Mittelschulen herausgegeben vom Verein schweizerischer Mathematiklehrer.) Zürich und Leipzig, Orell Füssli Verlag, 1938. 219 pages. fr. 3.80.

Archimedes. By E. J. Dijksterhuis. (Historische Bibliotheek voor de Exacte Wetenschappen, Deel VI, Eerste Deel.) Groningen-Batavia, P. Noordhoff, N.V., 1938. 213 pages. f 4.50.

Arithmetik. By Paul B. Fischer. (Sammlung Göschen, Band 47.) Berlin, Walter de Gruyter and Company, 1938. 152 pages. RM 1.62.

The Book of Time. By Gerald Lynton Kaufman. New York, Julian Messner, Inc., 1938. 16+287 pages. $3.00.

Calculus. By H. W. March and H. C. Wolff. Third Edition. (Modern Mathematical Texts, edited by C. S. Slichter.) New York and London, McGraw-Hill Book Company, 1937. 17+424 pages. $2.50.

Algebras and Their Arithmetics. By L. E. Dickson. Reprint of 1923 edition. New York, G. E. Stechert and Company, 1938. 12+241 pages. $2.50.

The Calculus. By R. D. Carmichael, J. H. Weaver and L. LaPaz. Revised Edition. Boston, Ginn and Company, 1937. 16+384 pages. $3.00.

REVIEWS

A New Geometry. By A. W. Siddons and K. S. Snell. Cambridge, The University Press, 1938. 16+304 pages. $1.32.

This new book is based on an older one by Godfrey and Siddons, but it is arranged according to an entirely original plan. It furnishes a complete course in geometry, but it would be better if it were preceded by a preliminary course of one year in length.

The first part of the book is intended to give the pupil a knowledge of geometrical facts together with developing the power to appreciate geometrical arguments and to use the methods of deduction. These aims are to be realized by means of worked examples and calculations, constructions, and originals. Theorems are arranged by groups in chapters; the results are stated and the proofs are left as originals. Questions on solid geometry are included and for convenience are grouped separately.

In view of all of the work that has been done in England in recent years to improve the teaching of geometry in the schools and in view of the standing of the authors, teachers of geometry everywhere will want to study the organization and methods of presenting geometric materials set forth in this new book. They will be particularly interested in seeing how plane and solid geometry are taught together in the same course.

W. D. Reeve

Introduction to the Theory of Equations. By Louis Weisner. New York, The Macmillan Company, 1938. 9+188 pages. $2.25.

In this distinctive text the author employs the concept of a field to effect the unification of the various topics of Theory of Equations. The result is a book in which the chapters are not separate entities but are related in a satisfactory manner.

The chapter titles are

I. Complex Numbers.
II. Division and Factorization of Polynomials in a Field.
III. Further Properties of Polynomials in a Field.
IV. Theory of Equations in the Field of Rational Numbers.
V. Theory of Equations in the Field of Real Numbers.
VI. Elimination. Resultants. Symmetric Functions.
VII. Algebraic Extensions of a Field.
VIII. Algebraically Closed Fields.
IX. Constructions by Ruler and Compasses.

It will be noted that chapters on determinants and systems of equations are not included. However, the author has compressed, without any apparent effort, an amazing amount of information in a book of this size. Much of this has been accomplished by more than 500 exercises satisfactorily distributed over the wide range from drill problems to extensions of theory. Readily accessible references, given with some exercises in the miscellaneous set of 115 in the back of the book, should stimulate the student. The style is pleasing, the explanations clear, and there are numerous illustrative examples.

The reviewer finds much to praise and little to criticise. Even in a brief review attention should be called to the generality with which Budan's Theorem and Descartes's Rule of Signs are stated, the elegance of Chapter VI, and the proof of the Fundamental Theorem of Algebra in Chapter VII.

There are a few minor points where the reviewer is in disagreement with the author. No careful distinction is made between equations and identities, the word "equation" being used for both and the conventional symbol for an identity is never used. The word "equation" is also used for an equality between constants. At the top of page 61 we find "multiplying the equation" rather than the more precise "multiplying the coefficients of the equation."

The few typographical errors noted are found on pages 15, 32, 46, 59, 64, 123. This carefully planned and well executed book should prove a welcome addition to a field that is certainly not overcrowded.

F. A. LEWIS

The Reverse Notation. By J. Halcro Johnston. London and Glasgow, Blackie and Son Limited, 1937. 10+74 pages. 3s. 6d.

In *The Reverse Notation* the author finds a place for proposals of reforms of all sorts, from the calendar and standards of weights and measures, to international finance. However, his chief argument, as well as his most convincing and that most likely to be of interest to the mathematicians, is for a change in our system of expressing numbers. This change is more sweeping than those usually proposed by advocates of a duodecimal system in that it combines the substitution of 12 for 10 as the basis of counting with the introduction of negative digits as well as positive.

The advantages of these negative digits are convincingly expounded. Subtraction is entirely eliminated and approximate additions, multiplications, and divisions are made easy since the neglect of the last few figures makes errors which tend to cancel each other rather than add up. In places where positive and negative numbers are likely to enter, as in the keeping of accounts, only one column is necessary to enter and sum numbers of both sorts. The practical advantages of using as a base a number rich in small factors are also given a detailed exposition. The arithmetic of the new notation is carefully explained with numerous examples, and tables of addition, multiplication, conversion from the present system, and logarithms are given.

Although the considerations advanced are of necessity mainly practical and have little bearing on mathematics, the mechanics of the reverse notation is interesting to explore for its own sake and the book certainly deserves the attention of all those interested in possible, though perhaps unattainable, improvements in our present system.

W. R. TRANSUE

Theory of Equations. By J. M. Thomas. New York and London, McGraw-Hill Book Company, 1938. 10+211 pages. $2.00.

In writing this textbook suitable for a term's course the author states in the preface that he "has been guided by a two-fold purpose: to make the treatment agree in spirit and terminology with what is called modern algebra, and to lead up to the Galois theory." Although courses in theory of equations have been offered regularly, no new text written in English has entered the field in over fifteen years. In view of the developments of modern algebra, it would appear that there is a decided need for just such a book. In fact, the reviewer is in sympathy with the author's purpose, and he exceedingly regrets that a critical reading of the book reveals significant imperfections.

In Chapter I a brief account of the number system is followed by elementary number theory. Properties of linear congruences are developed and used as a tool in succeeding chapters. Chapter II contains a discussion of permutations including a new proof of the theorem on the decomposition of a given permutation into transpositions. This will be of particular interest to teachers of group theory to whom the use of the alternating function in this connection seems artificial. The symmetric and alternating groups are defined, but the general definition of a permutation group is not given. Modern notation in Chapter III enables the author to give elegant proofs of some properties of determinants and matrices, the latter being written without the usual parentheses or double bars. The notation of Chapter III enables the author to give excellent proofs of the usual theorems on linear equations and ranks of matrices in Chapter IV.

In Chapter V entitled "Polynomials in a Single Indeterminate," the ordinary synthetic division appears (without proof) as well as a more general form. There are sections on the remainder and factor theorems, derivatives, multiple roots, highest common factor, reducibility, Gauss's Lemma and Eisenstein's Theorem. Chapter VI on "Graphical Methods" contains, in addition to theorems relating to polynomials, a brief treatment of complex numbers. There is also included a definition of continuity, accompanied by a few related exercises which the reviewer regards as essential for the student who is meeting the ϵ, δ language for the first time. Chapter VII contains an adequate discussion of roots of unity including theorems on the irreducibility of the cyclotomic and a binomial equation. The conventional symbol for an identity is used only in case of congruences.

The first seven chapters occupy almost the first half of the book, and Chapter VIII entitled "Single Equation in Single Unknown" takes up almost half the remainder. This chapter contains several interesting contributions to standardized material, among them being "the statement of Budan's and Sturm's theorems so as to include the upper end point of the segment considered" and "an elementary discussion of a limit to the error in Horner's method." Some teachers will regret that Newton's method is not included.

The last four chapters are brief but contain what should be considered ample treatment of the following topics, which are also the respective chapter titles:

Symmetric Functions, Constructibility, Resultants and Discriminants, and Simultaneous [non-linear] Systems.

The type is relatively large and well spaced. There are less than 400 exercises mostly of a routine nature supplemented by a miscellaneous set of 60 at the end of the book. It would seem that practically all of the material could be covered by advanced undergraduates in one term.

It is unfortunate that a book which otherwise is satisfactory should be marred by many careless slips on the part of the author and those who helped him see the book through the press. On page 59 the author defines "a *root* of the polynomial $f(x)$ and of the equation $f(x)=0$." While this is not objectionable *per se*, in the opinion of the reviewer this definition will increase the number of students who use the terms polynomial and equation synonymously. The definition may not be the "root" of all the following evils, but some statements from the text are offered to substantiate the reviewer's contention. On pages 96 and 99 the reader is asked to find equations having certain roots, but the answers are polynomials. On page 121 the reader is asked to locate the real roots of certain equations which are in reality polynomials. On page 188 he is asked to reduce certain systems of equations, but the systems in question are polynomials. On page 160 the reader is directed to substitute $\frac{y_4+y_5}{y_1+y_3}$ in x^3+x+1 and "clear of fractions." On page 114 we find "If these three equations are transposed to the left and multiplied together, a polynomial F of degree 3 results, ..."

In the interest of clearness, precision, and faultless style, some sentences should be rewritten. For example, on page 30 we find "Similarly for subscripts, that is, we ignore i as a superscript and j as a subscript." On page 54 the following sentence appears as a paragraph: "To emphasize the difference between them and indeterminates complex numbers will be called constants." On page 28 listed as an elementary transformation appears "(ii) multiplying every element in a fixed row by the same non-zero quantity and the determinant by its reciprocal." On page 157 in discussing the intersection of two circles the author writes: "If A-A' and B-B' are not both zero, this is the equation of a straight line passing through the points of intersection, that is, it is the equation of the common chord, and the intersections of the circles are its intersections with either circle." Certain other involved constructions may be found on pages 11, 19, 37, 42, 78, 121, 144.

In conclusion the reviewer would like to point out a few needed corrections of a somewhat different nature. The word "associated" is omitted on page 69. On page 75 the word "other" is needed in the sentence "If a circle be drawn with center at one of them, say a, and passing through the nearest of the others b, there is no root of the polynomial within the circle." The word "minus" should be omitted on page 106. Certain inequalities are reversed on page 122. Some mention should be made of convergence of infinite series on page 151.

F. A. LEWIS

Carl Friedrich Gauss—Inaugural Lecture on Astronomy and Papers on the Foundations of Mathematics. Translated and edited by G. W. Dunnington. Baton Rouge. Louisiana State University Press, 1937, 11+91 pages. $1.00.

This little book contains the substance of a lecture on the early life of Gauss, which Professor Dunnington gave the Kansas Section of the Mathematical Association of America in 1933 (see this MONTHLY, July 1933), and in addition English translations of a number of little known Gaussian fragments on the foundations of mathematics and on astronomy. The report on the early life of Gauss (pp. 1–32) deals with Gauss's activities from his boyhood (he was born in 1777) till 1803. Then follows a paper written by Gauss in his earlier years, which has only recently been brought to light (pp. 38–42, *Werke*, vol. XII, pp. 57–61); this contains eleven theses on the foundations of mathematics. Then follow a few questions on foundations, taken from Gauss's 19th manual, begun May 1809 (pp. 136–137), which follows directly after the paper dated Dec. 4, 1825, vol. VIII, p. 444 of Gauss's collected works (*Werke*, vol. X, pp. 396–397). The remaining part of the book is devoted to Gauss's inaugural lecture on Astronomy, probably written in 1807 (*Werke*, vol. XII, pp. 177–198). Professor Dunnington adds a number of explanatory notes to this as well as to the other papers of Gauss. Added is an interesting youth portrait of Gauss, from the Heimats museum at Göttingen.

Though this edition does not seem to present us with any startling discovery concerning Gauss's work, we may be grateful to Professor Dunnington, whose work allows a larger number of students in the English-speaking countries to obtain first-hand knowledge of some of Gauss's fundamental ideas. The report on Gauss's early activities is also especially welcome.

The book is dedicated, on the occasion of the bicentennial jubilee of the University of Göttingen, not only to Professors Hasse and Riesle, but also to the rector, dean and curator of this university "who today are ably and worthily guiding the destinies of the University of Göttingen."

D. J. STRUIK

Introduction to College Mathematics. By M. A. Hill, Jr. and J. B. Linker. New York, Henry Holt and Company, 1938. 373 pages + 93 pages of tables. $2.40.

This book is based upon First Year College Mathematics by the same authors, which has been reviewed by Frink.*

The earlier book consisted of parts: (I) Algebra and Trigonometry, (II) Analytic Geometry, and (III) Mathematics of Finance. The book under review is divided into Part I: Algebra and Trigonometry; Part II: Analytic Geometry and Calculus. Part I of the newer book is identical with the first part of the older one; the Analytic Geometry of the later book is, except for a slight rearrangement of topics and the omission of rotation of axes and a discussion of the conic in general position, the same as the second part of the first book. The Mathematics of Finance (90 pages) of the earlier book has been replaced by 54 pages of Calculus. Consequently we consider here only the section devoted to the Calculus.

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* This MONTHLY, vol. 44, 1937, p. 651.

Formulas are given for the differentiation of algebraic functions only. Applications include tangents and normals to curves, maxima and minima, and rates. There are numerous examples illustrating the processes involved, but, too frequently, these examples are used to give *rules* of procedure rather than to clarify concepts. Occasionally, the examples are used to give a "proof" by plausibility when a simple proof is available.

The discussion of differentials, the writer feels, would confuse a student. For instance, having defined dx as the differential of x, the reader is told to take dx equal to Δx (p. 316). Why and how is left an open question. Again, since the differential is a quantity defined in terms of a function $f(x)$ at $x=x_0$, corresponding to an increment Δx, Δf and df should be compared for this value of x. Whereas, in order to make $\Delta f=df$ (p. 317), Δf is evaluated at $x=x_0$ and df at $x_0+\frac{1}{2}\Delta x$. This hardly clarifies the distinction between differential and increment.

Integration is considered as the inverse of differentiation only. Since this is emphasized it seems strange that the formula for $\int x^n\,dx$ is not obtained from the corresponding differentiation formula instead of merely suggesting the plausibility of the general formula by verifying it for $n=1, 2, 3, 4$, (p. 336).

Applications of integration are made to the finding of areas, falling bodies, work, and momentum.

On the whole, a student should be able to read the book with understanding. Those instructors who are content to have their students learn rules of procedure in the elements of the Calculus should find this text fairly satisfactory.

C. A. NELSON

Procedures and Metaphysics. By E. W. Strong. A study in the philosophy of mathematical-physical science in the sixteenth and seventeenth centuries. Berkeley, University of California Press, 1937. 7 + 301 pages. $2.50.

This book is an important source of information on the motives which inspired the founders of modern mathematical-physical science. The author has carefully studied the prefaces and commentaries in which the great investigators and teachers of the sixteenth and seventeenth centuries considered the methods and nature of their inquiry. These texts afford a direct testimony by the scientists themselves concerning the character of their work, their opinions of the use of geometry in physical investigation and demonstration, the relation between pure and practical geometry and arithmetic, their ideas on points and lines, and their use in practical mensuration compared to the concepts of Euclid. Prominent in this research is one particular topic. In how far were these scientists influenced by philosophical, by metaphysical speculations, especially by the Neo-Platonist and Neo-Pythagorean thought of their days, with its idealization of mathematics?

The author did not raise this question without some provocation. Professor E. A. Burtt, in his book *The Metaphysical Foundations of Modern Physical Science* (Harcourt, Brace & Co., New York, 1925) has suggested that the Platonic and Pythagorean tradition, asserting the cosmological status of mathematics, has provided a foundation and a justification for modern science. "The Neo-Platonic background of the mathematical and astronomical development of the time of Galileo," asserts Burtt, "has strongly penetrated the mind of the Italian scientist, as in the case of so many lesser figures."

Mr. Strong's book aims at the refutation of Burtt's thesis. He insists that the metamathematical tradition which regarded the study of mathematics mainly as a preparation for the study of the mystical nature of the universe was without any considerable influence on the new mathematical and mechanical research. The roots of the work of Tartaglia, Cardan, Galileo, Kepler and the other productive mathematicians did not lie in the Platonic tradition, but in the study of Euclid, or more general, in the operational approach exemplified by Euclid. Euclidean geometry offered their scientists a scientific structure which first establishes the meaning of its fundamental concepts and then develops by strictly logical procedure its consequences without any appeal to metaphysical speculations.

The task is accomplished in the following way. First we have an exposition of the function of mathematics in the Platonists and Pythagoreans of antiquity, as Nicomachus, Theon, Proclus. These opinions are compared to those professed by Tartaglia, Cardan, Cataneo, and other Italian and French mathematicians in the sixteenth and seventeenth centuries as Vieta, Clavius, Leonardo da Vinci, and others. Then comes an investigation of the status of the science of mechanics in Italy prior to Galileo, followed by procedures and metaphysics in Galileo and Kepler. The final chapter deals with the metamathematical tradition in the early modern period, and discusses the Platonic schools of Ficino, Pico, Reuchlin and Bungus with their influence upon mathematical writers as Zamberti, Domenico, Henrion and Dee. An appendix on the operational meaning of point and line in Euclid's *Elements* presents us with a discussion on the opinions of Mach and Poincaré. Strong's conclusion is: "Euclid offers us no evidence that he began his geometry by first entering into a theory of knowledge. . . . If Euclid's geometry needs any foundations for the meaning of its definitions, the subject-matter and procedures of the geometry itself would appear to be a sufficient basis. We do not need to appeal to the so-called more ultimate grounds of mind and nature. . . . A task of operational method is to keep clear the domain of reference intended by the questions we ask. Questions of relevancy or meaning are kept straight when we do not confuse our categories. fields, contents or universes of discourse. No geometrician needs metaphysical cement to connect pure and practical geometry" (pp. 243–244).

According to Strong, Galileo and the other productive mathematicians and physicists of his period are examples of the operational approach to science. They do not assert that "mathematical demonstrations in science must be based on metamathematical doctrines or principles." "The methodological position brought out in the study of the Italian investigators and the analysis of the methodological insight in Kepler culminating in the *Epitome*," he concludes, "constitute an appreciable weight of evidence for the thesis that operational considerations existed in the early modern period without being based upon metamathematical foundations and without requiring subsequent metaphysical arguments, distinctions, and sanctions" (p. 183).

Kepler shows more influence of the metaphysical tradition, but Mr. Strong gives ample evidence that Kepler, in places where this tradition came in conflict with the evidence of an operational approach, does not hesitate to desert the metaphysical speculation for a scientific reasoning in the sense of Euclid.

Mr. Strong's case is very good, but it seems to the reviewer that he has gone somewhat too far in his attempts to separate the two traditions. In Kepler's case he mainly bases his conclusions upon the *Epitome*, one of Kepler's later books (1620), but there is a wealth of material in Kepler's work which shows how deeply his productive work was stimulated by his Platonic way of thinking. He was always looking for geometrical and arithmetical harmonies, because of his deep conviction of the metaphysical importance of mathematics. The cogency of a scientific hypothesis never was for him independent of metaphysical beliefs. We see in Kepler more than in Galileo the transition from scientific reasoning influenced by platonical metaphysics to a more "operational" way of thinking.

We also think of Copernicus, or Nicolaus Cusanus, not mentioned by Mr. Strong, as cases where metaphysical reasoning influenced the productive use of mathematics in physical study. We may even take Giordano Bruno, who has not lent his name to any particular theory or theorem in science, but who has influenced further mathematical and physical thought to a considerable extent.

It is certain, however, that Mr. Strong has shown that we cannot simply assume that the platonic revival of the sixteenth century is the cause of the approval of mathematical methods in the subsequent study of mechanics and physics. There was a wide gap between the two traditions. The question now takes a new form, which Mr. Strong has not attempted to investigate in his book. Why did philosophy begin to platonize in the early modern period? Why did, shortly afterwards, mathematics also enter into the study of nature? The answer seems to lie in a study of the social and economic forces of the period. The great social and technological changes affected both the philosophers and the scientific workers. University and practical work, however, were largely separated. Men like Tartaglia and Stevin had their education even entirely outside of the university halls. We seem to have currents due to similar causes in fields rather widely separated. Where university and practice joined hands, however, the platonic tradition let its force be felt, largely as an antidote to the aristotelian tradition. For this reason we still must take it into consideration as a secondary factor in the development of modern natural science.

D. J. STRUIK

General Mathematics. A One Year Course. By Harris Crandall and F. Eugene Seymour. New York, D. C. Heath and Company, 1937. 8+389 pages. $1.28.

This book presents a "bird's eye view of elementary mathematics" and is designed to appeal to the "natural interests" of Junior II and Senior I High School students including the "non-mathematical" pupils. A student will do much valuable mathematical sight-seeing in the journey mapped out. An intelligent student will so enjoy the glimpse as to want further travel—and too late in his school course to get it—while the slow one should pause for a better view. Both will miss significant "sights" and experiences, even the cartesian graph of a simple equation.

While "undue emphasis has not been placed on any one of the three fields—arithmetic, algebra, and geometry"—a better interrelation of these fields would have helped to clinch them all. Algebra misses the chance provided by the excellent exercises on pages 8 and 9 to stand out as an enlarged, handy form of arithmetic, where numbers (*e.g.* lengths of lines) are not required to tell their magnitude, where old words like "product" and "factor" occur in their old meanings and new terms, like "coefficient," are helpfully introduced. The application of axioms to simple equations is not closely enough tied in with their use in demonstrative geometry and the reviewer could wish for a sharper distinction between a "rule" and an "axiom" than is likely to follow from Unit IV. The use of the circle graph as an exercise on the central angle is one of the clever correlation touches which makes one wish for more.

Student interests are cleverly met, even in the strategically placed reviews. Exercises and tests show variety and practical interest, particularly in the excellent Unit of Indirect Measurement in which simple applications to Physics are wisely included.

Some trouble-making errors are noted. For example, on page 65, a student, asked to "supply the word necessary to make each of the following statements completely true," will meet these statements:

"11. To bisect a line a perpendicular is drawn to it at its . . .
12. To bisect an angle we must find a point . . . from its arms."

The statement *0*12, repeated on p. 67, line 22, is particularly misleading, as the concept of "distance from a point to a line" has not been considered and a false notion could only too naturally result, on the basis of a construction practiced and not proved.

The book is not bulky, its pages are pleasing, and its striking sketches catch the mind as well as the eye. An intriguing cover design demands the immediate use of a straight edge to check the faulty guessing of the eye.

The book should prove of value in many classes, especially if more than a year can be allowed, so that its meat may be supplemented with some less concentrated material and its well-chosen topics be fully appreciated, even if no additional principles can be introduced.

ELIZABETH M. COOPER

MATHEMATICS CLUBS

EDITED BY E. H. C. HILDEBRANDT, New Jersey State Teachers College

All reports of club activities, suggestions, topics with references, and other material of interest to clubs should be sent to E. H. C. Hildebrandt, New Jersey State Teachers College, Upper Montclair, N. J.

TOPICS FOR CLUBS

The following subjects have appeared on a number of club programs and if bibliographies could be sent to us for them, they will perhaps furnish additional material for discussions and papers.

31. Continued fractions.
32. Maps and map projection.
33. Constructions with compass alone.
34. Constructions with ruler alone.

CLUB REPORTS, 1937–1938

Pi Mu Epsilon, Washington University

At the regular meetings the following topics were discussed by members of the faculty: The aesthetics of science; The application of mathematics to meteorology; Problems in modern dynamics; Steiner's ellipse and point; reviews of *Men of Mathematics* and *Mathematics for the Million*. An open meeting was held in conjunction with the local chapter of Sigma Xi with Dr. John von Neumann of the Institute for Advanced Study speaking on "Foundation Problems of Quantum Mechanics," and "Foundation Problems of Mathematics." At the initiation banquet, Dean A. S. Langsdorf spoke on "The Method of Inversion as Applied to Electrical Problems."

Director, Dr. Gabriel Szegö; Vice-Director, Bernice Dunie; Secretary, Candace Wisbrock; Treasurer, Prof. E. Stephens.

Pi Mu Epsilon, Oklahoma A. and M. College

The organization, formerly the Mathematics Club, was installed last year as a chapter of Pi Mu Epsilon, with twenty charter members. The papers presented during the year were: Cryptographs; Solution of equations of the fourth degree; The mathematical meaning of infinity and some infinite collections; Why nine is a magic number; The weather; Japanese mathematics; Complex numbers; Mathematics for aerial photography; Prime numbers; The slide rule; Mathematical fallacies.

Director, Wilma Meacham; Vice-Director, C. E. Abraham; Secretary-Treasurer, Charles G. Cruzan; Sponsor, Dr. E. F. Allen.

Mathematics Club, Stanford University

The Mathematics Club as an organization attended the two lectures at Stanford by Dr. T. C. Fry of the Bell Telephone Laboratories, (1) On locating roots of polynomials by matrix iteration, and (2) Applications to network design. The second talk was accompanied by a motion picture of a machine for calculating the roots of polynomials in operation. Such subjects as Researches in higher geometry; An elementary proof of R. Hall's theorem on soluble groups; Lower bounds for the degree of multiply transitive group in terms of its degree of transitivity; A geometric proof of Minkowski's theorem on the product of two non-homogeneous linear forms in two variables; On the duration of play; On the luminosity of stars; were the basis of papers and discussions during the year.

Executive Council, Dorothy Evans, D. Belle Rundle, Edward Bewley; Adviser, C. D. Olds.

Pi Mu Epsilon, Hunter College

The chapter held eight program meetings, an initiation banquet in the fall with Professor Courant of New York University as guest speaker, an initiation tea in the spring with Professor Mayme I. Logsdon of the University of Chicago as guest speaker, and a reunion supper in June.

The subject for the first semester was geometrical topics *via* complex numbers. The first two meetings concentrated on geometry growing out of the addition of complex numbers, the last two on geometry growing out of the multiplication of complex numbers. The reports of the second semester dealt with algebraic plane curves. A meeting each was devoted to singular points and lines, poles and polars, Cremona transformations, the theorem of Noether.

Prizes were awarded to the best speakers. The awards for the first semester were won by Estelle Poppiani and Helene Parnass, and for the second semester by Jeannette Miller.

Director, Prof. Marguerite D. Darkow; Vice-Director, Rita Mohnkern; Recording secretary, Wilma Szabo; Corresponding secretary, Selma Goldstein; Treasurer, Pearl Goldman.

Mathematics Club, Hunter College

The year's program was planned to include topics from the mathematics of statistics, since a number of the club members were particularly interested in that field. Dr. Helen Walker of Columbia University addressed the club on "Statistical concepts which should be familiar to every educated person." Other papers presented: Calculus of variations; A game of logic; Some aspects of the Social Security Program; Algebra of logic; The four color problem; The trisection of an angle; A curve of learning; Some statistical measures; A geometric interpretation of the coefficient of correlation; The application of complex numbers to the solution of geometric problems; Puzzles and fallacies.

In addition to several social meetings, the annual party given by the club to all students of the department included for entertainment the play *It Can't Happen Here* written by Dr. Whelan of the department faculty.

Two competitive examinations were held. An integration contest was won by Malene Anderson, and a special mathematics examination with a prize of twenty-five dollars was won by Helene Parnass.

President, Lillian Leight; Vice-President, Vivian Fruchtbaum; Secretary, Justine Schmertz; Treasurer, Bessie Hochberg. Advisers, Dr. Jewell H. Bushey, Mrs. Helen Kutman.

The Mathematical Club, Regis College

This club sponsored a contest for secondary school students on applications of mathematics. The six prize winners were guests of the club at a spring tea. Papers read at the meetings were: Construction of a beehive; Amicable numbers; Different radices; Women in mathematics.

President, M. Melanson; Vice-President, M. White; Secretary, M. Gahan; Treasurer, E. Mahoney.

Sigma Phi Mu, New Jersey State Teachers College

The club observed the memorial year in honor of Dr. Slaught in a twofold way. Professor Davis, a former student of his, spoke on the life of this modern mathematician, and the famous play by Dr. Slaught *The Evolution of Numbers* was presented. Each of the classes in the club presented a mathematical play suitable for presentation before high school students. They were: Mock trial of B versus A; The evolution of numbers; Flatlanders; The clue of the little 3. Topics discussed at the remaining meetings were: Extracting cube roots mentally; Interesting sidelights on mathematics; Properties of the catenary; Logarithmic spirals in nature; Magic squares; Crystallography; Famous Americans as mathematicians; Mathematical induction; Mathematics in navigation; Repeating decimals. One of the guest speakers was ninety-five-year-old Dr. William Vail, author of *Div-A-Let,* who spoke on Division by letters, and on the Perpetual calendar.

President, Norman Chinoy; Vice-President, Marie Albers; Secretary, Marjorie MacInnes; Treasurer, Sheva Crystal; Adviser, Dr. E. H. Hildebrandt.

Kappa Mu Epsilon, University of New Mexico

This chapter cooperates with several other institutions in the Southwest, in sponsoring an annual lecture given by a member of the Southwestern Section of the Mathematical Association. Dr. Wexler of Arizona State College was the traveling lecturer last year. One monthly program consisted of a talk on properties of numbers and the other meetings included talks on the life of the mathematicians: François, Viète Archimedes, Gauss, and Descartes.

President, Louis York; Vice-President, Clara May Mathew; Secretary, Osborn Keller; Treasurer, Charles Barker; Adviser, Dr. H. D. Larson.

Mathematics Club, University of Alberta

Papers presented before this club are entered in competition for the Dr. Cook prize for the best student paper. It was awarded to Allen Gibb for his presentation of the calculus of variations. Other papers were: Design by geometry; Mathematics of insurance; Graphical representation; Projective coordinates; The discriminating cubic; The philosophy of Dr. A. N. Whitehead. One meeting was devoted to the solution of a set of problems distributed earlier in the year, and involving a bit of mathematical brain teasing. At the annual banquet, Professor Keeping spoke on the book *Philosophy and the Physicists* by Dr. Susan Stubbings.

President, D. R. Crosby; Secretary, Marjorie Stockwell; Treasurer, L. Palleson; Adviser, Dr. A. J. Cook.

Euclid's Circle, Mount St. Scholastica College

The main topic for discussion for the year's programs included a study of the contributions of the various nations to the fields of mathematics. Other topics were: The life and work of Euclid; Culture and mathematics; and a review of *Mathematics for the Million.* Part of each meeting was given over to trick problems, mathematical games and puzzles. One meeting featured the play *A Problem for Pythagoras.*

President, Miriam Powers; Vice-President, Mary Schirmer; Secretary-Treasurer, Mary Donahoe; Sponsor, Dr. A. J. Reardon.

Mathematics-Physics Club, College of Saint Theresa

Reports presented by faculty and student members included: Science of musical sounds; Beauty of a triangle; Teaching the subtraction of signed numbers; Mathematics in Germany; A billion; The spider problem. One meeting commemorated the Geometry of Descartes. At another a great deal of interest was aroused in a debate on the subject: The study of mathematics should have practical applications as its sole aim. Social meetings were held before Christmas and at the close of the year.

President, Mary Payant; Vice-President, Mary McKeown.

Pi Mu Epsilon, University of California

Papers presented during the year were: Uniformly almost periodic functions; Secondary schools and secondary teachers of mathematics in England and France; The isoperimetric problem; The application of mathematics to the earthquake problem; Certain quadric, cubic and quartic loci in space of three dimensions; Mathematical games; Mathematical economics; An application of the complex variable to a geometric problem; Selected topics from the theory of relativity; Oddities of mathematics.

Director, Raymond Wakerling; Vice-Director, Virginia Wood; Treasurer, Charles Hayes; Secretary, Margaret Seegmiller.

PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, H. L. OLSON, AND W. F. CHENEY, JR.

ELEMENTARY PROBLEMS

Send all communications concerning Elementary Problems and Solutions to W. F. Cheney, Jr., Dept. Box 35, Connecticut State College, Storrs, Connecticut.

The Department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 362. *Proposed by V. Thébault, Le Mans, France.*

If X and Y are consecutive positive integers, and Z is an integer, and if $X^2+Y^2=Z^2$, show that each of the two numbers, $(X+Y\pm Z)^2+1$, is the sum of the squares of two consecutive integers, and that of these last four integers the two odd ones are squares, and the even ones are the doubles of squares.

E 363. *Proposed by D. L. MacKay, Evander Childs High School, N. Y.*

Construct triangle ABC, given A, a, and (h_a+c-b).

E 364. *Proposed by A. V. Richardson, Bishop's College, Lennoxville, Quebec.*

If $(1+x)^n/(1-x)^3=a_0+a_1x+a_2x^2+\cdots$, show that

$$a_0+a_1+a_2+\cdots+a_{n-1}=\frac{n}{3}(n+2)(n+7)\cdot 2^{n-4}.$$

E 365. *Proposed by Virgil Claudian, Bucharest, Rumania.*

Prove that

$$\int_0^1 x^n(1-x^n)^n dx=\frac{n^n(n!)}{(n+1)(2n+1)(3n+1)\cdots[(n+1)n+1]}.$$

E 366. *Proposed by C. O. Oakley, Haverford College.*

Two ferry boats ply back and forth across a river with constant speeds, turning at the banks without loss of time. They leave opposite shores at the same instant, meet for the first time 700 feet from one shore, continue on their way to the banks, return and meet for the second time 400 feet from the opposite shore. As an oral exercise, determine the width of the river.

E 367. *Proposed by Cezar Coşniţă, Focşani, Roumania.*

The point P moves on the circumcircle of triangle ABC, and the bisectors of angles APB and APC meet AC and AB at Q and R respectively. Show that QR passes through the center of the circle inscribed in triangle ABC. Show also that if PS and PT are perpendicular to PQ and PR and cut AC and AB at S and T respectively, then ST passes through the center of the escribed circle which touches side BC between B and C.

E 368. *Proposed by W. B. Campbell, Drexel Institute.*

A clock has its hour, minute, and second hands turning about the common center, and all together at noon. They will not be all three coincident again for another twelve hours, although there will be several earlier instants when two of the three hands are superimposed. At which of these instants will the remaining hand make the least angle with the two which are superimposed?

E 369. *Proposed by A. V. Richardson, Bishop's College, Lennoxville, Quebec.*

Find the form of n if $1^4+2^4+3^4 \cdots +n^4$ is exactly divisible by $1^2+2^2+3^2+ \cdots +n^2$.

E 370. *Proposed by V. Thébault, Le Mans, France.*

Locate the point P within the irregular tetrahedron $ABCD$ so that each of the six planes, each through P and an edge, will bisect the surface of the tetrahedron.

SOLUTIONS

E 321[1938, 185]. *Proposed by V. Thébault, Le Mans, France.*

In what system of enumeration may a four-digit number of the form *aabb* be the square of a number of the form *cc*, provided furthermore that if expressed in the decimal system, a and b consist of the same digits in opposite order?

Solution by C. W. Trigg, Los Angeles City College.

The given data may be expressed as $ar^3+ar^2+br+b=(cr+c)^2$, or as $a(r-1)+(a+b)/(r+1)=c^2$. Also, $a=10m+n$ and $b=10n+m$, with m and n each less than ten, a and b less than r, and r greater than ten.

But $(a+b)/(r+1)=11(m+n)/(r+1)$ must be an integer, and since $1 \leqq m+n \leqq 18$, the eligible values of r are restricted to twenty-four, namely, 11, 12, $\cdots$, 17, and $11k-1$, where $1<k\leqq 18$. The latter group yields no solutions, since $a<r$. The further restrictions imposed by c^2 limit the solutions to $r=43$, with $40\ 40\ 04\ 04=(41\ 41)^2$, and $04\ 04\ 40\ 40=(13\ 13)^2$, or $r=65$, with $15\ 15\ 51\ 51=(31\ 31)^2$, or $r=109$, with $28\ 28\ 82\ 82=(55\ 55)^2$, or $r=131$, with $48\ 48\ 84\ 84=(79\ 79)^2$.

Also solved by E. P. Starke and the proposer.

E 322[1938, 185]. *Proposed by J. E. Trevor, Cornell University.*

A manufacturer makes all possible sizes of brick-shaped blocks of molded building material, which are such that the lengths of the edges are integral multiples of the unit of length, and that the number of units in the total length of the twelve edges of a block is equal to two-thirds of the number of units of volume in the block. How many and what sizes does he make?

Solution by E. P. Starke, Rutgers University.

Let the numbers of units of length in the three edges meeting at a vertex be x, y, and z. Then, by hypothesis, we have

$$4x + 4y + 4z = 2xyz/3,$$

or

$$6(x + y + z) = xyz.$$

Evidently one letter is a multiple of 3. Let x be that one, and put $x=3u$. Then $6u+2y+2z=uyz$, or

$$u = 2(y + z)/(yz - 6). \tag{1}$$

Thus $u=2(yz+y^2)/[y(yz-6)]=(2w+2y^2+12)/yw$, where $w=yz-6$ is a divisor of $2y^2+12$. If we put $y \leqq z$, (1) implies $yz-6 \leqq 2(y+z) \leqq 4z$, or $y \leqq 4+6/z \leqq 10$. For assigned values of $y=1, 2, 3, \cdots, 10$, possible values of $w=yz-6$ may be easily determined as divisors of $2y^2+12$. If the corresponding z and u are integers, we have a solution. In this way we find eight essentially distinct solutions for (x, y, z), namely (48, 1, 7), (27, 1, 8), (12, 1, 13), (9, 1, 20), (18, 2, 4), (6, 2, 8), (12, 3, 3) and (3, 4, 7).

Also solved by W. E. Buker, W. B. Clarke, Michael Goldberg, A. V. Richardson, C. W. Trigg and the proposer.

E 323[1938, 185]. *Proposed by J. R. Musselman, Western Reserve University.*

If D, E and F are the mid-points of the sides BC, CA and AB respectively of the triangle ABC, show that the centers of the nine-point circles of triangles AFE, BDF and CED form a triangle homothetic with triangle ABC in the ratio 1:2.

Solution by A. V. Richardson, Bishop's College, Lennoxville, Quebec.

For the respective triangles, ABC, AFE, BDF and CED, denote the nine-point centers by K, L, M and N, the orthocenters by O, P, Q and R, and the circumcenters by S, T, U and V.

It is clear that since K, L, M and N bisect respectively the segments, OS, PT, QU and RV, and since the triangles AFE, BDF and CED are each similar to triangle ABC and of one-half its linear dimensions, then $KA=2KL$, $KB=2KM$, and $KC=2KN$. Hence the triangle LMN is homothetic with the triangle ABC, the ratio 1:2, and the homothetic center K.

In his solution, E. S. Smith of Boulder, Colorado, points out that K is also the nine-point center of triangle LMN, and investigates the center of symmetry for the congruent triangles DEF and LMN. If the circumcenter and orthocenter of triangle LMN are denoted respectively by H and J, he finds S, H, K, J and O equally spaced in that order on a straight line. If the common mid-point of HK and SJ is denoted by G, he finds that G is the center of perspective of triangles DEF and LMN.

Also solved by W. B. Clarke, T. C. Esty, D. L. MacKay and C. W. Trigg.

E324[1938, 185]. *Proposed by W. F. Cheney, Jr., Connecticut State College.*

Right triangles with integer sides proportional to 3:4:5 are well known. The sides of such a triangle form an arithmetic progression. Show that these constitute the only right triangles having integer sides in arithmetic progression, and furthermore that no right triangle exists having integer sides in geometric progression or in harmonic progression.

Solution by Fred Discepoli, New York, N. Y.

If $0<d<a$, and $a-d$, a and $a+d$ are the sides of a right triangle, then $(a-d)^2+a^2=(a+d)^2$. This reduces to $a(a-4d)=0$. The only admissible solution here is $a=4d$, and the sides of the triangle are $3d$, $4d$ and $5d$, and hence in the required ratio.

To investigate the possibility of a right triangle having sides in geometric progression, let the sides be a, $\sqrt{ab}$ and b, so that $a^2+ab=b^2$. This gives the solution, $a=b(\sqrt{5}-1)/2$, which cannot be satisfied if a and b are both integers.

To investigate the possibility of a right triangle having sides in harmonic progression, let the sides be $1/(a+d)$, $1/a$ and $1/(a-d)$. The Pythagorean relation, $1/(a+d)^2+1/a^2=1/(a-d)^2$, reduces to the quartic equation, $a^4-4a^3d-2a^2d^2+d^4=0$. Since this does not factor rationally, a is not expressible rationally in terms of d, and hence $a=d$ times an irrational expression. So no triangle can have integer sides in harmonic progression.

Note by the Proposer. If the sides of a right triangle are in arithmetic progression, they can only be in the ratio of 3:4:5, even though they be irrational. If the sides are in geometric progression, they must be in the ratio $1:r:r^2$, where r^2 is the limit of the ratio of any term to its successor in the Fibonacci series, 1, 1, 2, 3, 5, 8, 13, $\cdots$. If the sides are in harmonic progression, two shapes are possible, with incommensurable sides, since the equation, $x^4-4x^3-2x^2+1=0$, has two positive, irrational roots and two imaginary roots.

Also solved by E. F. Allen, Lois E. Bell, J. H. Edmonston, H. L. Lee, M. Y. Luke, D. L. MacKay, A. V. Richardson, E. P. Starke, C. W. Trigg, Paul Weisz and the proposer.

E 325 [1938, 185]. *Proposed by W. W. Bigelow, Beloit College.*

Show that the radius of the sphere which can displace the most liquid from a filled conical wineglass is given by the formula, $r=hn/(n^2+n-2)$, where h is the altitude of the cone, and n is the cosecant of its generating angle.

Solution by D. L. MacKay, Evander Childs High School, New York, N. Y.

If x is the altitude and y the radius of the base of the segment of the sphere protruding above the glass, then $y^2=2rx-x^2$, $h-r+x=rn$, $x=r(1+n)-h$, and $dx/dr=1+n$.

The displaced volume, $V=4\pi r^3/3-\pi xy^2/2-\pi x^3/6=\pi(8r^3-3xy^2-x^3)/6=\pi(8r^3-6rx^2+2x^3)/6$. The volume V is a maximum when

$$24r^2-6x^2-12rx\cdot dx/dr+6x^2\cdot dx/dr=24r^2-6x^2-12rx(1+n)+6x^2(1+n)=0,$$

or when $4r^2-2rx(1+n)+x^2n=0$. But this factors into

$$(2r-x)(2r-nx)=0; \qquad \text{so} \qquad r=x/2 \qquad \text{or} \qquad nx/2.$$

When $r=nx/2$, since $x=r(1+n)-h$, $r=hn/(n^2+n-2)$ and $V=(4\pi h^3/3)\div(n-1)(n+2)^2$, the maximum displaced volume. When $r=x/2$, $r=h/(n-1)$ and $V=0$. Then the sphere is above the liquid and tangent to its surface. This case does not satisfy the condition that the sphere touch the glass.

E. P. Starke calls attention to the fact that this problem appeared in Granville's *Elements of the Calculus*, 1911, p. 118, with the answer in the form, $r=h\sin\alpha/(\sin\alpha+\cos 2\alpha)$.

Also solved by W. E. Buker, J. H. Edmonston, V. E. Pound, S. A. Singer, Alvin Spiro, R. H. Urbano and the proposer.

Note by the Editor. A full discussion of this problem was recently given by W. R. Longley in a note entitled "An example of a continuous function with finite discontinuities in its second derivative," this MONTHLY, vol. 44, 1937, pp. 467–470.

E 326[1938, 185]. *Proposed by J. Rosenbaum, Bloomfield, Connecticut.*

Find all positive integer solutions, with $x<y$, of $x^2+y^2+xy=20461$.

Solution by H. T. R. Aude, Colgate University.

Write $x^2+y^2+xy=f(x,\ y)$. It is seen that $f(1,\ 2)=7$, $f(3,\ 4)=37$, and $f(3,\ 7)=79$, and that $20461=7\cdot 37\cdot 79$. It is readily verified that the product of $f(a,\ b)$ by $f(c,\ d)$ has the two forms, $f(bc-ad,\ ad+bd+ac)$ and $f(bd-ac,\ ac+bc+ad)$. It follows that the product, $f(1,\ 2)\cdot f(3,\ 4)$ is equal to both $f(2,\ 15)$ and $f(5,\ 13)$. Again the product formulas with the use of the identities, $f(-a,\ b)=f(a,\ b-a)$, and $f(a,\ b)=f(b,\ a)$, show readily that the products $f(3,\ 7)\cdot f(2,\ 15)$ and $f(3,\ 7)\cdot f(5,\ 13)$ yield respectively, $f(31,\ 125)$ and $f(65,\ 99)$, and $f(4,\ 141)$ and $f(76,\ 89)$. Consequently it is possible to write

$$20461 = f(1,\ 2)\cdot f(3,\ 4)\cdot f(3,\ 7) = f(31,\ 125) = f(65,\ 99) = f(4,\ 141) = f(76,\ 89).$$

Hence the required solutions $(x,\ y)$ are $(4,\ 141)$, $(31,\ 125)$, $(65,\ 99)$ and $(76,\ 89)$.

Also solved by W. E. Buker, Wm. Douglas, A. V. Richardson, E. P. Starke, C. W. Trigg and the proposer.

E 310[1938, 47]. *Proposed by V. Thébault, Le Mans, France.*

In a certain system of notation there exists a four-place number of the form $aabb$ which is the square of bb. Show that the numbers, b and $a^2+4(a-1)^2$, are each perfect squares. Determine the base of such a system and the values of a and b, knowing that a is also a perfect square.

Partial Solution by Irene Price, State Teachers College, Oshkosh, Wisconsin.

Let B be the base of the number system for which $aabb=(bb)^2$. Then

$$aB^3 + aB^2 + bB + b = (Bb + b)^2.$$

Dividing by $(B+1)^2$, which is not zero, we obtain

$$b^2 = \frac{aB^2 + b}{B + 1} = a(B - 1) + \frac{a + b}{B + 1}\cdot \tag{1}$$

Since b is an integer, $a+b$ must be a multiple of $B+1$. But a and b are each less than B, so $a+b=B+1$, and $b=B+1-a$. Substituting this value for b in (1), we have

$$(B + 1 - a)^2 = a(B - 1) + 1. \tag{2}$$

The resulting quadratic equation in B must have rational roots; so its discriminant, $5a^2-8a+4=a^2+4(a-1)^2$, must be a perfect square, x^2. Similarly, since $5a^2-8a+4-x^2=0$ has just one positive integer root a, which equals $[4+\sqrt{5x^2-4}]/5$, we see that $(5x^2-4)$ must be a perfect square y^2. Then $a=(4+y)/5$, and $y=5a-4$. Substituting the squares of the successive integers for a, we find that the first which satisfies all requirements is $a=16$. Then $y=76$, $x=34$, $B=40$, $b=25$, and the number $aabb=16\ 16\ 25\ 25=(25\ 25)^2$. Moreover, $b=5^2$ and $a^2+4(a-1)^2=34^2$ are perfect squares.

Editorial Note. No proof has been received that b is always a square.

Also partially solved by G. W. Wishard and W. E. Buker.

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well known textbooks or results found in readily accessible sources will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTIONS

3903. *Proposed by Simon Mowshowitz, New York, N. Y.*

Denote $[(n!)!]!$ by $n(!)^3$, etc., $n(!)^0=n$. Prove that for $k\geqq 2$

$$\frac{n(!)^k}{(n!)^{[n-1]![n!-1]![n(!)^2-1]!\cdots[n(!)^{k-2}-1]!}}$$

is an integer.

3904. *Proposed (July 1937) by the late R. P. Baker, University of Iowa.*

ABC is a given triangle; find the condition that a point P may be constructed in the plane of ABC such that

$PA:PB:\ PC = p:\ q:\ r$, (p, q, r real positive constants).

3905. *Proposed by V. Thébault, Le Mans, France.*

A hexagon $(H)\equiv A_1A_2A_3A_4A_5A_6$ with its opposite sides parallel is inscribed in a circle (O), and the orthogonal projections of a point M of its plane on its sides A_1A_2, A_2A_3 etc. are B_1, B_2, B_3, B_4, B_5, B_6 forming in this order a hexagon (H'). Show that (1) the sides of (H') in sets of three are parallel to two determined directions; and that (2) the locus of the points M such that the area of (H') is constant is a circle concentric with (O), and conversely.

3887 [1938, 482]. *Correction.* The equation in the last line should be $\rho_1+\rho_2 = r\sec^2\theta$.

SOLUTIONS

3810 [1936, 643]. *Proposed by Oystein Ore, Yale University.*

One may define a new "multiplication" in the system of all positive real numbers by putting

$$[a, b] = a^b.$$

Determine all positive *rational* numbers for which this multiplication is:

I) *Commutative* $[a, b]=[b, a]$.
II) *Associative* $[a, [b, c]]=[[a, b], c]$.
III) *Right-hand* or *left-hand distributive*

$$[(a + b), c] = [a, c] + [b, c], \quad [c, (a + b)] = [c, a] + [c, b].$$

Solution by E. P. Starke, Rutgers University.

(I) If $b=a$, we have an obvious solution. Otherwise take $b>a$ and put $b=ra$ to get $a^{r-1}=r$ or $a=r^{1/(r-1)}$, $r>1$. Put also $1/(r-1)=u/v$, where u and v are integers with no common divisor $\neq 1$. Since a and r are rational, $r=(v+u)/u$ must be an exact v-power. Then u and $v+u$, being relatively prime, are integral v-powers. But any two integral v-powers have a greater difference than v unless $v=1$. Hence $1/(r-1)=u$ is an integer. Then $a=[(u+1)/u]^u$ and $b=[(u+1)/u]^{u+1}$.

(II) $a^{(b^c)}=(a^b)^c$ is evidently satisfied by $a=1$, b and c arbitrary, or by $c=1$, a and b arbitrary. If then $a\neq 1$, $c\neq 1$ we must have $b^c=bc$ or $b^{c-1}=c$. Thus by the above argument, if u is any integer $c=(u+1)/u$, $b=[(u+1)/u]^u$, a arbitrary.

(III) In $(a+b)^c=a^c+b^c$ without loss of generality we may take $a\geqq b$ and put $a=rb$. The given equation then reduces to $(1+r)^c=1+r^c$. For the moment consider r as varying continuously. Then put $f_1(r)=(1+r)^c$ and $f_2(r)=1+r^c$. The derivatives are $f_1'(r)=c(1+r)^{c-1}$ and $f_2'(r)=cr^{c-1}$. Thus for all positive values of r, $f_1'(r)$ is greater than, equal to, or less than $f_2'(r)$ according as $c>1$, $c=1$ or $c<1$, respectively. But $f_1(0)=f_2(0)$. Hence $f_1(r)$ is greater than, equal to, or less than $f_2(r)$ according as $c>1$, $c=1$ or $c<1$. Thus the complete solution is given by $c=1$, a and b arbitrary.

In $c^{a+b}=c^a+c^b$ we may take $a\geqq b$. Let $a=br$ and put $c^b=v/u$, $r=p/q$, with

p, q, u, v positive integers and each fraction in its lowest terms. Also $p \geqq q$. The given equation can then be written $v^{p-q}(v-u)^q = u^p$. If $v=1$, then $(1-u)^q = u^p$. Since $p \geqq 1$, $q \geqq 1$, every prime factor of u is a divisior of $1-u$, hence of 1. Thus $u=1$, but $u=1$ does not satisfy $(1-u)^q = u^p$. Hence $v \neq 1$. Since v is not a divisor of u^p, we must have $p-q=0$. But then $b=a$ and $v-u=u$, or $u=1$, $v=2$. Thus the complete solution is given by $a=b=1/n$, $c=2^n$, n any integer.

We consider now the determination of sets of elements such that the given property of the multiplication in each case is true for any subset of the elements in a chosen order and where a given element may be repeated. As shown above for (I) the set has only one element, which is arbitrary; or the set has two distinct elements as indicated.

For (II) suppose first that the set has an element unity and an element $a \neq 1$. Then from $[[a, 1], a] = [a, [1, a]]$ we have $a^a = a$; and therefore $a=1$. Hence there is only one element, unity. Suppose no element is unity, then from $[a, [a, a]] = [[a, a], a]$ we have $a^a = a^2$ and $a=2$. Thus there is only one element in the set, either 1 or 2.

For III(a) $(a+a)^a = 2a^a$, or $2^a = 2$. Hence $a=1$; and 1 is the only element in the set. For III(b) $a^{(a+a)} = 2a^a$ or $a^a = 2$. Since this is impossible for a rational a, the set is empty.

Solved also by W. Penney and J. Rosenbaum.

Editorial Note. Penney stated that (I) was treated by H. L. Slobin in this MONTHLY, 1931, p. 444.

3811 [1937, 54]. *Proposed by N. A. Court, University of Oklahoma.*

A plane parallel to the base ABC of the tetrahedron $DABC$ meets the edges DA, DB, DC in the points P, Q, R; the same edges are met in the points U, V, W by a plane antiparallel to the plane ABC. Show that, if the plane PQR varies, the radical axis of the three spheres $AQRV$, $BRPW$, $CPQU$ remains fixed.

Note. For a definition of antiparallel see Court, *Modern Pure Solid Geometry*, p. 247. The Macmillan Co., 1935. The corresponding problem in the plane was considered in *Educational Times*, Reprints, vol. 60, 1894, p. 107, Q. 12001.

Solution by the Proposer.

The two lines QR, VW are antiparallel with respect to the angle BDC; hence the four points Q, R, V, W are concyclic. Similarly for the points R, P, W, U and for P, Q, U, V.

The spheres $(B) = BRPWU$, $(C) = CPQUV$ have the points P, U in common. On the other hand the four points V, W, B, C being concyclic, the point $X = (BW, CV)$ has equal powers with respect to the spheres (B), (C), hence the plane $XUP = XDA$ is the radical plane of these two spheres.

Likewise, if $Y = (CU, AW)$, $Z = (AV, BU)$, the planes YDB, ZDC are the radical planes of the pairs of spheres (C), $(A) = AQRVW$; (A), (B). Now if $X' = (DX, BC)$, $Y' = (BY, CA)$, $Z' = (CZ, AB)$, the three lines AX', BY', CZ' have a point L in common, namely, the trilinear pole, with respect to the triangle ABC, of the trace of the plane UVW in the plane ABC. Hence the three planes $DAXX'$, $DBYY'$, $DCZZ'$ have the line DL in common, this line is therefore the radical axis of the three spheres considered. Now the line DL is determined by the tetrahedron $DABC$ and the plane UVW; hence the proposition.

3812 [1937, 55]. *Proposed by N. A. Court, University of Oklahoma.*

The vertex of a variable cone is fixed, the base being the circle of intersection of a fixed sphere with a variable plane passing through a fixed straight line. The cone cuts the sphere in a second circle: show that the plane of this second circle passes through a fixed line.

Solution by the Proposer.

Let (S) be the given sphere, V the fixed vertex, and (p) the circle of intersection of (S) with the plane (P) passing through the given fixed line a. The line of intersection m of the plane (P) with the plane (Q) of the second circle, which the cone $V-(p)$ and the sphere (S) have in common, lies in the polar plane Σ of V with respect to (S).

The traces b, x of the fixed plane $V-a$ on the planes Σ, (Q) pass through the fixed point $L=(am)=(a, \Sigma)$. The flat pencil of rays a, b, x, LV in the fixed plane $V-a$ is harmonic, and the three lines a, b, LV are fixed, hence the fourth ray, x, is also fixed, hence the proposition.

Note. An analogous problem in the plane was considered in the *Nouvelles Annales de Mathématiques*, 1882, p. 426, Q. 1381.

Solved also by R. Seamons and C. E. Springer.

3813 [1937, 55]. *Proposed by V. Thébault, Le Mans, France.*

Given a circle (O) and two orthogonal axes xx' and yy'; a variable point P on (O) is projected orthogonally into M and N on xx' and yy'; then, on the perpendicular from P to MN, two points V and W are taken so that the lengths of VP and PW are each in a given ratio k with the distance MN. (1) On VW as diameter a circle (Σ) is described. Show that the radical axis of any two circles (Σ) passes through a fixed point. (2) Determine the envelope of the circles (Σ). (3) Examine the special case $k=1$.

Solution by C. E. Springer, University of Oklahoma.

Let xx' and yy' be taken as axes for rectangular coordinates, with xx' passing through the center of the given circle (O). Then the coordinates of P, a variable point on the fixed circle $(x-a)^2+y^2=r^2$ may be taken as $(a+r\cos\theta, r\sin\theta)$. The variable circle (Σ) has the equation

$$(1) \qquad [x-(a+r\cos\theta)]^2+[y-r\sin\theta]^2=k^2\sigma^2,$$

where $\sigma^2=\overline{MN}^2=a^2+r^2+2ar\cos\theta$. Writing this equation with $\theta=\theta_1$ and then with $\theta=\theta_2$ and subtracting, we find the equation of the radical axis of the two circles to be

$$[x - a(1 - k^2)] \sin (\theta_1 + \theta_2)/2 - y \cos (\theta_1 + \theta_2)/2 = 0,$$

which passes through the fixed point $[a(1-k^2), 0]$. Eliminating θ between (1) and its derivative with respect to θ, we find the envelope of the circles (Σ) to be an Oval of Descartes with the equation

$$(2) \qquad [x^2 + y^2 - 2ax + (1 - k^2)(a^2 - r^2)]^2 = 4k^2r^2(x^2 + y^2).$$

In case $k=1$, the radical axis of any two circles (Σ) passes through the intersection of xx' and yy', and the equation of the envelope of the circles (Σ) reduces to

$$[x^2 + y^2 - 2ax]^2 = 4r^2(x^2 + y^2).$$

On writing this in polar coördinates one obtains

$$(3) \qquad \rho = 2(a \cos \theta \pm r)$$

from which the envelope can be readily constructed.

Some interesting properties of the Ovals of Descartes may be found in "*Traité des Courbes Spéciales Remarquables Planes et Gauches*" by F. Gomes Teixeira, Tome 1, pp. 218–233.

Editorial Note. The proposer stated the above results for the radical axis, and that the envelope of the circles (Σ) in the general case is an oval of Descartes with F, the intersection of the axes x, y, as one focus: no proof of this last result was given. We shall add some remarks on the solution and consider the other foci. It is simpler to write (2) in the polar form

$$(2) \qquad \rho^2 - 2a\rho \cos \theta + (1 - k^2)(a^2 - r^2) - 2kr\rho = 0.$$

In obtaining this equation we get also another with the last term positive; but the two equations give the same curve and we discard the second. Hence in (3) we may omit the minus sign. Let F and F' be a pair of foci for an oval of Descartes; and set $FF'=d$, $FP=\rho$, $F'P=\rho'$. Then, if λ and μ are constants, the curve is defined by the equation $\lambda\rho+\rho'=\mu$. Taking F as the pole and FF' as the axis, we obtain the equation of the curve in polar coordinates

$$(4) \qquad (\lambda^2 - 1)\rho^2 - 2\rho[\mu\lambda - d \cos \theta] + \mu^2 - d^2 = 0.$$

The identification of (2) and (4) and the solution of the resulting three equations gives two cases for F'

$$F_1': \quad d = (1 - k^2)a, \qquad \lambda = \pm k, \qquad \mu = \mp r(1 - k^2),$$

$$F_2': \quad d = (a^2 - r^2)/a, \qquad \lambda = \pm r/a, \qquad \mu = \mp kd.$$

For (3), where $k=1$, the foci F and F_1' coincide, but for $r \neq a$ we have the focus F_2'. If also $r=a$ we obtain the cardioid, for which all three foci coincide.

For the case $k=1$ it is quite easy by synthetic geometry to determine the form of the envelope.

3814 [1937, 55]. *Proposed by I. J. Schoenberg, Colby College.*

Let $f(t)$ be a one-valued and complex-valued function which satisfies the functional relation

$$e^{iks}f(t) = f(t+s) - f(s) \tag{1}$$

for all real values of t and s, where k is a real constant $\neq 0$. Prove that

$$f(t) = C(e^{ikt} - 1)/ik \qquad (C \text{ a constant}). \tag{2}$$

Remark. It should be noticed that for $k=0$ (1) and (2) reduce to (1′) $f(t+s)=f(t)+f(s)$ and (2′) $f(t)=Ct$ respectively. It is known that (1′) implies (2′) only if some additional assumption is made on $f(t)$, for instance boundedness in the neighborhood of some point (see G. Hamel, Mathematische Annalen, vol. 60, 1905, pp. 459–462). No such assumption is needed if $k\neq 0$.

Solution by Gaines B. Lang, Georgia School of Technology

Multiplying both sides of (1) by e^{ikt}, we obtain

$$e^{ik(t+s)}f(t) = e^{ikt}f(t+s) - e^{ikt}f(s). \tag{3}$$

Interchanging s and t in (1) we obtain

$$e^{ikt}f(s) = f(t+s) - f(t). \tag{4}$$

Combining (3) and (4) we have

$$e^{ik(t+s)}f(t) = e^{ikt}f(t+s) - f(t+s) + f(t) \tag{5}$$

or

$$[e^{ik(t+s)} - 1]f(t) = [e^{ikt} - 1]f(t+s).$$

Hence:

$$\frac{f(t)}{e^{ikt} - 1} = \frac{f(t+s)}{e^{ik(t+s)} - 1}.$$

Since the last equation is an identity, and since one member depends only on t while the other depends only on $(t+s)$, it is clear that each member must be a constant. Hence (2) results.

Solved also by M. T. Bird, B. P. Hoover, O. J. Ramler, R. A. Rosenbaum, C. E. Springer and the proposer.

3815 [1937, 110]. *Proposed by P. Turán, Budapest, Hungary.*

Given the function of a real variable x

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{x}{4^n};$$

show that there is a positive constant c_1, independent of x, such that

$$|f(x)| < c_1 \log\log x, \qquad x > e. \tag{1}$$

Show also that there exists a sequence $x_1 < x_2 < \cdots \to \infty$, and a positive constant c_2, independent of x, such that

$$(2) \qquad |f(x_\nu)| > c_2 \log\log x_\nu, \qquad \nu = 1, 2, \cdots .$$

Solution by the Proposer.

Proof of (1). We have

$$f(x) = \sum_{n \leqq [\log x]} + \sum_{n > [\log x]} = S_1 + S_2.$$

$$(a) \qquad S_1 < \sum_{n \leqq [\log x]} \frac{1}{n} < 2 \log\log x.$$

$$(b) \qquad S_2 < \sum_{n > [\log x]} \frac{x}{n 4^n} < x \sum_{n > [\log x]} \frac{1}{4^n} < c_3.$$

From (a) and (b) part (1) follows.

Proof of (2). Set $x_\nu = (4^\nu - 1)2\pi/3$, then $x_\nu/2\pi$ is an integer; and, if $\mu \leqq \nu$,

$$(c) \quad (4^\nu - 1)/3 = 1 + 4 + \cdots + 4^{\nu-1} \equiv 1 + 4 + \cdots + 4^{\mu-1} \equiv (4^\mu - 1)/3, \bmod 4^\mu.$$

$$(d) \qquad f(x_\nu) = \sum_{\mu=1}^{\infty} \frac{1}{\mu} \sin \frac{4^\nu - 1}{3} \frac{2\pi}{4^\mu} = \sum_{\mu=1}^{\nu} + \sum_{\mu=\nu+1}^{\infty} .$$

For the first sum in (d) we have by (c)

$$(e) \qquad \left| \sum_{\mu=1}^{\nu} \frac{1}{\mu} \sin \frac{4^\mu - 1}{3} \frac{2\pi}{4^\mu} \right| = \left| \sum_{\mu=1}^{\nu} \frac{1}{\mu} \sin \left(\frac{2\pi}{3} - \frac{2\pi}{3 \cdot 4^\mu} \right) \right| > c_4 \log \nu;$$

and for the second sum we have

$$(f) \qquad \left| \sum_{\mu=\nu+1}^{\infty} \frac{1}{\mu} \sin \frac{4^\nu - 1}{3} \frac{2\pi}{4^\mu} \right| < \frac{4^\nu - 1}{3} 2\pi \cdot \frac{1}{4^{\nu+1}} \cdot \frac{4}{3} < \frac{2\pi}{9} \cdot$$

Since $\nu \sim \log x_\nu / \log 4$, if $\nu \to \infty$, we have at once from (e), (f), (d) the desired result for (2).

Editorial Notes. Since $f(x)$ is an odd function we may take $x > 0$ in the consideration of $|f(x)|$. For the proof of (a) we have the elementary inequalities

$$\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} < \log n < 1 + \frac{1}{2} + \cdots + \frac{1}{n-1},$$

where $n \geqq 2$ for the left side inequality. The inequality in (a) is not true unless $n \geqq 3$ and $x \geqq e^3$. If these latter inequalities are true, then $1 \leqq \log n$ and (a) is then true; and we may take $c_1 = 2.1$.

In (f) the terms of the sum are all positive and hence $f(x_\nu) > (\sqrt{3}/2) \log \nu$, $x_\nu < 4^\nu(2\pi/3)$. It is convenient to consider first $\nu \geqq 2$, and then $\log x_\nu < \nu[\log 4 + 2^{-1} \log (2\pi/3)] < 1.76\nu$. We then have

$$\log \nu > \log \log x_\nu / [1 + \log 1.76/\log 2].$$

This gives

$$f(x_\nu) > 0.47 \log \log x_\nu,$$

which is also true for $\nu = 1$.

3816 [1937, 110]. *Proposed by E. Weiszfeld, Budapest, Hungary.*

Given the function of the complex variable z

$$f(z) = \sum_{i=1}^{n} \frac{a_i - z}{|a_i - z|},$$

where a_i, $(i = 1, 2, \cdots, n)$, denotes any given complex number; show that $|f(z)|$ takes on its maximum value in any domain not containing the points a_i at the boundary of the domain.

Solution by P. Erdös, Budapest, Hungary.

The theorem is equivalent to the following vector-theorem.

Let n points $A_1, A_2, \cdots, A_n$ be given in the plane, and P an arbitrary point: we define the vector

$$\overrightarrow{f(P)} = \sum_{i=1}^{n} \frac{\overrightarrow{PA_i}}{\overline{PA_i}}$$

For any domain not containing the points $A_1, A_2, \cdots, A_n$, the function $\overrightarrow{f(P)}$ takes the maximum of its absolute value at the boundary of the domain.

Let us suppose that the theorem is not true. Then there is in the interior of the domain a point O for which $|\overrightarrow{f(O)}|$ is maximum.

Let R be the endpoint of $\overrightarrow{f(O)}$. Take on the straight line OR a point Q lying in the interior of the domain such that O lies between Q and R. We prove that in contradiction to our supposition

$$|\overrightarrow{f(Q)}| > |\overrightarrow{f(O)}|. \tag{1}$$

Take with O as origin a rectangular coordinate-system such that the vector $\overrightarrow{f(O)}$ lies in the direction of the positive x axis. The projections of any vector v upon the x and y axis we denote by v_x and v_y respectively. We have evidently

$$\overrightarrow{f(O)} = \sum_{i=1}^{n} \frac{\overrightarrow{OA_i}}{\overline{OA_i}}, \qquad \overrightarrow{f_x(O)} = \sum_{i=1}^{n} \left(\frac{\overrightarrow{OA_i}}{\overline{OA_i}} \right)_x. \tag{2}$$

But

$$\left(\frac{\overrightarrow{QA_i}}{\overline{QA_i}} \right)_x > \left(\frac{\overrightarrow{OA_i}}{\overline{OA_i}} \right)_x, \qquad (i = 1, 2, \cdots, n), \tag{3}$$

since the length of each vector is 1 and the first vector forms with the x axis a smaller angle than the second. Hence by addition

$$\overrightarrow{f_x(Q)} = \sum_{i=1}^{n} \left(\frac{\overrightarrow{QA_i}}{\overline{QA_i}}\right)_x > \sum_{i=1}^{n} \left(\frac{\overrightarrow{OA_i}}{\overline{OA_i}}\right)_x; \tag{4}$$

from which by (2)

$$|\overrightarrow{f_x(Q)}| > |\overrightarrow{f(O)}|. \tag{5}$$

But since the vector cannot be smaller than its projection we obtain

$$|\overrightarrow{f(Q)}| > |\overrightarrow{f(O)}|,$$

the required inequality.

Remark. By the same method, we may prove the following general theorem:

Let $\phi_1(x), \phi_2(x), \cdots, \phi_r(x)$ be positive, monotonically increasing continuous functions defined for any real positive numbers. Then the function

$$F(z) = \sum_{i=1}^{n} \frac{a_i - z}{|a_i - z|} \phi_i(|a_i - z|)$$

takes the maximum of its absolute value for any domain not containing the complex numbers $a_1, a_2, \cdots, a_n$ at the boundary of the domain.

Editorial Note. In the theorem of the problem, but not in the Remark, the case where all the n points lie on a straight line should be excluded. For, if A_1 and A_n are the extreme points, we may select a region cutting the segment MN from the straight line so that all its points lie in the region on the same side of segment A_1A_n. Then for each point of MN we have the maximum value n. If we make this exclusion, then in (3) there is at least one value of i for which the given sign $>$ must be used; and in no case can $<$ be used as appears in the proof.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending news items to R. G. Sanger, Eckhart Hall, University of Chicago, Chicago, Illinois.

Professor B. P. Reinsch of Florida Southern College has been elected president of the Florida Academy of Sciences.

Professor J. A. Shohat of the University of Pennsylvania during the spring of 1939 is giving a series of six lectures on interpolation, approximation, and mechanical quadrature at the Graduate School of the Department of Agriculture in Washington, D. C.

At Brooklyn College, Professor L. T. Moore and Thomas Nicholson have been granted sabbatical leaves of absence for the year 1938–39. A. W. Landers, Jr., has been granted a sabbatical leave for the spring term, 1939. The following appointments have been made in the department of mathematics for the year 1938–39; Dr. Aaron Fialkow, Jeanette Fox Keston, Dr. Moses Richardson, and W. J. Van Stockum.

G. E. Jones, formerly supervisor of schools of Bienville Parish, has been appointed to an assistant professorship at Louisiana Polytechnic Institute.

Dr. P. S. Wagner, professor of mathematics at Lebanon Valley College for the past twelve years, died December 5, 1938.

THE COWLES COMMISSION FOR RESEARCH IN ECONOMICS

The Cowles Commission for Research in Economics will hold its Fifth Annual Research Conference on Economics and Statistics at Colorado College, Colorado Springs, U.S.A., from Monday, July 3 to Friday, July 28, 1939. The purpose of the Conference is to provide research workers an opportunity to present their problems and results before a group qualified to contribute constructive discussion. With the formal sessions limited to the morning hours, the atmosphere is considered more conductive to leisurely and thoughtful discussion than it is at the crowded programs of the regular sessions of learned societies.

The Cowles Commission Research Conferences originated in a series of informal sessions during the summer of 1935 after the meeting of the Econometric Society at Colorado Springs on June 22–24 of that year. At these gatherings various papers were presented and discussed by economists who remained in the vicinity. The meetings were so successful that it was decided to continue them in subsequent years. The scope of the Conferences has grown steadily. In 1935, the first year, there were 8 papers presented by 7 different lecturers, the total number attending being 25 of whom 5 were from out-of-town. Last summer 38 papers were presented by 27 different lecturers, 192 individuals, of whom 93 were from out-of-town, attending part or all of the sessions. Participants have come from all parts of the United States and from 16 foreign countries.

Among the speakers have been R. G. D. Allen, Louis H. Bean, Harold T. Davis, Edward L. Dodd, Griffith C. Evans, Mordecai Ezekiel, Arne Fisher, Irving Fisher, R. A. Fisher, Ragnar Frisch, Thornton C. Fry, Corrado Gini, Frank L. Griffin, Harold Hotelling, Edward V. Huntington, Emil Lederer, Wassily Leontief, A. P. Lerner, Alfred J. Lotka, Jakob Marschak, Karl Menger, James Harvey Rogers, Charles F. Roos, René Roy, Henry Schultz, Walter A. Shewhart, Carl Snyder, Gerhard Tintner, Abraham Wald, Ralph J. Watkins, S. S. Wilks, Elmer J. Working, Holbrook Working, and Theodore O. Yntema.

Each year the Cowles Commission publishes a report of the meetings, occupying about 100 pages, which includes abstracts of all papers presented. These reports are distributed to several thousand leading economists and statisticians throughout the world.

There is no charge for attendance at the Conference. All serious students are welcome. Room and board for those attending, and for their husbands or wives, will be available in dormitories of Colorado College at about $40 for the four weeks or $11 per week. Those interested should notify the Cowles Commission, 301 Mining Exchange Building, Colorado Springs, Colorado, U.S.A., in order that they may receive in the spring a preliminary announcement of the program.

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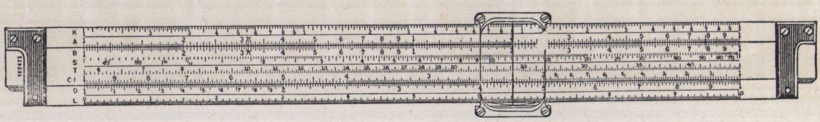

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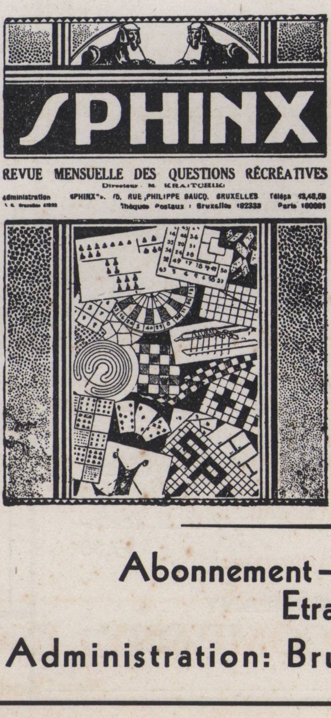

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